The Repayment of Local and Other Loans

Sinking Funds

BY

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PREFACE.

In writing this book my primary object has been to render it practically useful to those engaged in connection with the loan debt of public authorities and privately owned undertakings. It is the result of actual experience extending over many years, but the problems which have arisen have been isolated examples and have not occurred in any regular sequence, but often at long intervals. For this reason each problem has been treated independently in order that any particular adjustment may be made by reference to a similar example fully worked out and described. This entails a certain amount of repetition in order that each chapter may contain a brief summary, and a reference to the results, of previous and subsequent investigations. In order to carry this out in its entirety, each adjustment has been reduced to a series of stages briefly stated, and is accompanied by detailed statements of the method adopted, and actual calculations upon standard forms which I have specially prepared. But I have gone further than this, and have grouped the problems under the heads of the various factors governing a sinking fund, namely, the amount in the fund, the period of repayment, the rate of accumulation and the rate of income to be received upon the present investments. This orderly arrangement of isolated practical examples has compelled a theoretical method of treatment which has been adopted throughout. In my own earlier experience I very soon realised that even after making the fullest use of the published tables of compound interest, the ordinary methods of arithmetic were utterly inadequate and that all calculations must be made by logarithms; consequently a full knowledge of the use of a log. table has been assumed. The next practical difficulty which arose was that the ordinary published tables of compound interest very often did not contain the required rate per cent., and it therefore became necessary to make the calculation by other means. For this reason I have included in the earlier chapters a brief summary of the mathematical principles, showing the derivation of the formulæ upon which all the tables are constructed. In order, however, to render this method by formula generally available I have reduced all such calculations to simple rules, and in Chapter X, dealing with the

various standard calculation forms, minute instructions are given for finding the actual values of the whole of the factors relating to any rate per cent. I have, in fact, endeavoured to state the methods in such a manner that a knowledge of the meaning of the formula is not required.

Throughout the book the final results are expressed in decimal form. This course has been adopted partly in order to save time, but primarily to make the book applicable to any currency of a decimal nature; and it is recommended that all pro forma accounts should be prepared in this form.

The methods of adjustment described as the deductive method, the annual increment (ratio) method, and the annual increment (balance of loan) method, are not new. The terms have merely been used for convenience of reference.

The actual compilation of the book has occupied considerable time, and it has not been written in the consecutive order in which it is now presented to the reader. It has involved the preparation of many standard forms and statements, and very many calculations which are not given in the text. Every effort has been made to ensure absolute accuracy of detail, but in a few cases the final decimal figure in the result obtained may not agree with the result found by another method, and the same applies to the final figure of some of the logs. These small differences are not, however, of any practical importance. The cross-references to other results have been carefully compared, but they are very numerous and a few errors may be found.

The methods adopted, and the results obtained, have been in all cases verified by an alternative method of proof, even where it is not shown in detail in the text. The summaries of methods, and the rules and formulæ given in italics at the head of the various chapters have been in all cases carefully compared with the individual statements and with each other, and a uniform wording has, as far as possible, been adopted throughout.

As stated in the Introduction, no attempt has been made to include anything in the nature of a full statement of the statutory obligations as to the repayment of the loan debt of local authorities or the policy of Parliament with relation thereto, except so far as they affect the actual method of repayment.

In the final chapters dealing with the life of the asset, the equation of the period of repayment and the incidence of taxation a serious attempt has been made to elucidate this

difficult subject. It has been treated mainly from the mathematical standpoint, both as regards the annual instalment and interest upon the loan. It is a matter about which there is naturally some divergence of opinion as to its practical application, but I hope that the considerations here set forth will be of assistance in arriving at a definite solution.

The object of the book is entirely practical. I hope it will materially assist those who are approaching the consideration of such problems for the first time, and that it will also be useful to those who are already fully acquainted with the subject, if only because it attempts to deal with it in a consecutive and orderly manner.

In conclusion, I wish to tender my thanks to my friend and colleague, Mr. Arthur Holme, A.C.A., who has assisted me in the revision of the manuscript and the correction of the proofs, and also to my son, Edward Gordon Turner, who has rendered me very valuable aid in the preparation of the final manuscript and the verification of the various calculations in the text.

E. H. T.

Manchester,
November, 1912.

TO AMERICAN READERS.

In the early part of the year 1906, I was appointed the British Accounting Expert to the National Civic Federation of New York in the Enquiry into the Municipal and Private Ownership of Public Utilities (Municipal Trading) in Great Britain.

The enquiry was made by a Commission of Twenty-one, under the Chairmanship of Melville E. Ingalls, Chairman of the Board of Directors, Big Four Railroad, Cincinnati. The details of the enquiry were under the supervision of Dr. Milo R. Maltbie, now a Member of the Public Service Commission of the State of New York. The Report of the Commission is contained in three volumes, issued in New York in 1907.

The enquiry in Great Britain was confined to the results of municipal and private ownership and operation of gas works, street railways or tramways, and electric lighting and power undertakings. In the United States the enquiry was extended to water works.

The Commission examined 24 public and private undertakings in London, Birmingham, Dublin, Glasgow, Leicester, Liverpool, Manchester, Newcastle-on-Tyne, Norwich and Sheffield.

The following is a full list of the Committee on Investigation:—

*Melville E. Ingalls, *Chairman* (Chairman Board of Directors, Big Four Railroad), Cincinnati.

Albert Shaw, *Vice-Chairman* (Editor "Review of Reviews"), New York City.

Talcott William (Editorial Writer, the Press), Philadelphia.

W. D. Mahon (President Association Street Railway Employes), Detroit.

*Professor Frank J. Goodnow (Columbia University), New York City.

*Walton Clark (Third Vice-President The United Gas Improvement Company), Philadelphia.

*Professor Edward W. Bemis (Superintendent Water Works), Cleveland.

*Professor John H. Gray (University of Minnesota), Minneapolis.

Walter L. Fisher (Special Traction Counsel for City of Chicago and ex-President Municipal Voters' League), Chicago.

*Timothy Healey (President International Brotherhood Stationary Firemen), New York City.
*William J. Clark (General Manager Foreign Department, General Electric Company), New York City.
H. B. F. Macfarland (President Board of Commissioners, District of Columbia), Washington.
Daniel J. Keefe (President International Longshoremen's Association), Detroit.
*Professor Frank Parsons (President National Public Ownership League), Boston.
*Professor John R. Commons (Wisconsin University), Madison, Wisconsin.
*J. W. Sullivan (Editor "Clothing Trades' Bulletin"), New York City.
*F. J. McNulty (President International Brotherhood of Electrical Workers), Washington.
*Albert E. Winchester (General Superintendent City of South Norwalk Electric Works), South Norwalk, Conn.
*Charles L. Edgar (President The Edison Electric and Illuminating Company), Boston.
*Dr. Milo R. Maltbie (member of the Public Service Commission), New York City; now a member of the Public Service Commission of the State of New York.
Leo S. Rowe (University of Pennsylvania), Philadelphia.
*Edward A. Moffet, *Secretary* (Editor "Bricklayer and Mason"), Indianapolis, Ind.

The investigation in Great Britain was made by the members of the Committee indicated by an asterisk.

The following summary of the financial details of the undertakings examined will give some idea of the magnitude of the enquiry.

Loan Capital:—		
Outstanding:		
Municipal	£18,866,648	
Private	6,144,945	
		£25,011,593
Repaid by means of Sinking Fund:		
Municipal		3,868,301
		£28,879,894
Share Capital or Stock, Private		12,146,504
Total Capital raised		£41,026,398

Capital expended on Works:—

Municipal	£24,517,370	
Private	18,266,368	
		£42,783,738

Gross Revenue for the year under Examination:—

Municipal	£6,091,821	
Private	3,956,936	
		£10,048,757

The examination by the experts was commenced early in March, 1906, and continued without interruption until the end of July, 1906. The members of the Commission were engaged in Great Britain from 30th May to the end of July, 1906.

The expert assistance was divided between the Technical and Accounting sides. In each class of undertaking an American and a British technical expert made a joint investigation, valuation, and report.

The British technical experts were:—

Gas Works, Mr. William Newbigging, Consulting Engineer, of Manchester.

Tramways (Street Railroads), Mr. J. H. Woodward, Consulting Engineer, London.

The American technical experts were:—

Electricity Supply, Mr. Albert E. Winchester, General Supt. City of South Norwalk Electric Works, South Norwalk, Conn.

Gas Works, Mr. J. B. Klumpp, Inspecting Engineer of the United Gas Improvement Company, Philadelphia.

Tramways (Street Railroads), Mr. Norman McD. Crawford, Consulting Engineer, Hartford, Conn.

The accounting side of the whole of the undertakings examined was entrusted to myself and Mr. R. C. James, the Chief Accountant to the United Gas Improvement Company, of Philadelphia. During the investigation all the experts were in constant communication and worked together, and each undertaking was, as far as possible, examined by them at the same time. This resulted in a very minute comparison of American and British technical and accounting practice. During the months we were associated many friendships were formed, but

the greater value of the enquiry to all the experts engaged is undoubtedly the wider appreciation of American and British practice. On the technical and practical side, many hitherto diverging views were reconciled or reduced to an agreed mean, but as regards the accounting and financial side no such difference was found. In America, as in Great Britain, all problems relating to large financial operations follow the same economic laws, and the only variations on each side of the Atlantic are due to the actual conditions now existing. On this side the growth of large municipal undertakings, concerned with the provision of public utilities, has been a gradual process extending over many years requiring the constant supervision of Parliament. On the other side the municipal operation of public utilities is still in its infancy, and is therefore an urgent practical problem, as evidenced by the enquiry, made at very considerable expense, by the National Civic Federation of New York.

In connection with pure methods of accounting, it must be acknowledged that as regards uniformity the British system is not so far advanced as the American. Standard methods have there long been adopted for the accounts of gas works, electric lighting undertakings, and street railways, and include not only the form of the final Revenue Account and Balance Sheet, but also methods of analysis and book-keeping in minute detail. It is true that in Great Britain standard forms of accounts for gas works are contained in the Gas Works Clauses Act. The Board of Trade have, under their power in the Electric Lighting Acts, prescribed standard forms of account for electric lighting undertakings. A standard form of account for tramways has been drawn up by the Tramway Institute and the Institute of Municipal Treasurers and Accountants, and as the result of an enquiry by a Government Departmental Committee (report dated 1907) further amended standard forms have been suggested for all the above-mentioned utilities. The British standard forms, however, are deficient to the extent that they lack that minute attention to detail which is the characteristic of the American systems. A standard form of revenue, or profit and loss, account may be very useful for comparing the final results of the year's operations, but unless there is absolute uniformity of detail in the items charged to each head of operating expense it is impossible to make any reliable comparison between the operating costs of two or more undertakings. This fact was very clearly shown in the enquiry by the Commission in Great Britain.

Turning now to the immediate subject of this book, namely, the repayment of the loan debt of local authorities and privately owned undertakings, the result of the enquiry proved that as regards the actual methods of repayment the practice is the same in both countries. Another outstanding feature of the enquiry was the minute attention paid by the legal, labour and economic experts upon the Commission to the statutory and other obligations imposed by Parliament upon local authorities in Great Britain. This was due to the fact that the municipal operation of public utilities in this country is a much older institution than in the United States. The report of the Commission contains much valuable information in considerable detail as to these statutory obligations and is a useful resumé of a very complicated question, well worth perusal by municipal experts in this country. This book does not in any way attempt to deal with these matters in an exhaustive manner, and they are only mentioned in so far as they relate to the actual methods of repayment of loan debt. In writing the book I have borne in mind the results of the enquiry and have so arranged it that it will apply to all problems in whatever currency. The formula relating to a geometrical progression does not recognise any geographical limits, consequently the methods based thereon may be applied equally in the United States, as in Great Britain. Owing to the interest in municipal ownership and operation in the United States, the practical application of the statutory obligations as regards the repayment of loan debt by British municipalities, even as briefly stated, will, it is hoped, be useful.

As far as possible the terms used have been chosen in order to avoid any misunderstanding on either side. The word "Corporation" in Great Britain denotes both a Local Authority and a privately owned undertaking. The term "Local Authority" has therefore been used throughout, although it may not in a few cases be strictly correct. It includes all public authorities having control of public moneys for the public good and empowered to raise such moneys by way of an annual rate based upon an assessment of the annual value of the property. It is not the practice in Great Britain to levy a local rate or assessment based upon the capital value. All such rates are levied at so many pence in the pound (£) sterling of annual value.

In Great Britain there are several kinds of privately owned companies or corporations, which may be divided into two groups. First, those in which the liability of the members is

unlimited, which, however, are not numerous. Secondly, those companies in which the liability of the members is limited to the actual amount for the time being unpaid upon the shares held by them. Such companies may be divided into two classes, namely, those which derive their powers of operation from, and are incorporated by, special Act of Parliament; and, secondly, those incorporated under the Limited Liability Acts (The Companies Acts, 1862—1908). In this book all such companies have been included under the generic term "Private Undertakings" as distinguished from "Local Authorities."

The capital of all privately owned companies or undertakings in Great Britain is provided by the members and is raised in various ways. In the case of companies incorporated under the Limited Liability Acts the capital is invariably raised by the issue of a definite number of shares of a uniform nominal value, now as a general rule of £1 each. In the case of companies incorporated by special Act of Parliament the capital is sometimes raised by the issue of shares, similar to companies incorporated under the Limited Liability Acts, but often by the issue of stock. All capital raised by all privately owned undertakings, whether by shares or stock, may be issued with certain defined priorities or preferences both as to dividend and also as to repayment upon the final winding up of the company. In all cases considered, the capital provided by the members of a privately owned undertaking has been included under the generic term of "Share Capital or Stock."

The term "Loan Debt" has been used to denote all moneys borrowed by a public authority or private undertaking and secured by way of mortgage upon the assets, including in the term assets the power of a local authority to levy a rate or assessment. In the case of a private undertaking such loan debt is always repayable, on a winding up, in priority to the share capital or stock, and may or may not be repayable, during the life of the undertaking, by means of a sinking fund to be built up out of the annual profits. In the case of all public authorities in Great Britain, Parliament now invariably imposes the obligation to repay such loan debt by means of a sinking fund, or other alternative method, within a period having a more or less definite relation to the life of the asset created out of the loan, and such annual provision for repayment almost without exception operates immediately upon the borrowing of the money, and is charged against the annual rate or assessment levied by the local authority. In the case of a loan raised to provide works of a revenue-earning character, such as gas works,

etc., the annual redemption charges, as well as the interest upon the loan, are charged against the profits of the undertaking, and any deficiency is made good out of the annual rate or assessment levied upon the whole of the community.

The foregoing remarks will sufficiently explain the terms used in the book, but if fuller details are required the reader is referred to the Financial Appendix to the Report of the Commission (Vol. II, Part II, p. 628), prepared by Mr. James and myself.

E. H. T.

Manchester (Eng.),
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INTRODUCTION

CHAPTER I.

INTRODUCTION.

This book is the outcome of many years' professional work in connection with the accounts of municipal corporations, other local authorities, and privately owned commercial and financial undertakings. It deals only with the loan debt of such public authorities and private undertakings, and includes, in addition to the actual borrowing and repayment of the loan, the numerous problems which arise in connection with the Sinking Fund, relating to the amount in the fund, the rate of accumulation, and the period of repayment. The concluding sections contain chapters upon, (1) the relation between the life of the asset and the period of repayment; (2) the methods of finding the equated period of repayment; and (3) the equation of the incidence of taxation after the equation of the period, both as regards the annual instalment and interest upon the loan. In the last three chapters this difficult subject is treated in an exhaustive manner.

Subject Matter. The book does not pretend, in any way, to be a treatise upon the law relating to the loan debt of local authorities, or to give full particulars of the various statutory obligations, as regards repayment, imposed by Parliament; nor does it include a full statement of the general practice of Parliament and the government departments having control of such matters. All such statutory obligations are of a very variable nature and are contained in many general and special Acts of Parliament and provisional orders of the Local Government Board. The general practice cannot be said to be based upon any well-defined principles, but it should be stated that the Local Government Board endeavours, as far as it is empowered, to impose a uniform system, especially as to the periods to be allowed for the repayment of loans raised for public works having longer or shorter periods of duration or continuing utility.

Although the actual practice varies considerably in detail, the methods to be adopted in the solution of the various problems all follow certain well-defined mathematical rules; consequently the primary object is to demonstrate briefly the mathematical principles involved and afterwards to apply such

principles to a number of typical examples. As between the mathematical and practical sides of the subject, preference has been, and must necessarily be, given to the mathematical portion, because this is definite, and may be exactly stated. The practical variations from the ideal mathematical conditions are so numerous that only typical examples have been considered, although every effort has been made to include all the principal problems which are likely to arise.

MATHEMATICAL PRINCIPLES. Since all problems, relating to the future redemption or repayment of a present loan, to be spread over a period of years, involve questions of compound interest, it is first necessary to investigate the mathematical principles governing the annual or other periodic accumulation of a present sum of money, and also of a sum of money payable or receivable at the end of each of a number of equal and definitely recurring periods of time. All such problems follow the algebraical rule relating to a geometrical progression, as distinguished from the rule relating to an arithmetical progression which concerns only simple interest. The primary object therefore is to convert the simple algebraical formula relating to a geometrical progression into a formula which may be applied to the subject matter of this book, namely, compound interest. Simple interest is purely a matter of arithmetical calculation and does not arise in any way in the problems to be discussed. On the other hand, compound interest involves a mathematical method of calculation and affects all problems which will be hereafter considered.

The algebraical formula relating to any geometrical progression, as regards the last and first terms in any series of numbers is:—

$$l = ar^{n-1}$$

and this formula may be converted into a standard formula relating to the accumulation of a sum of money now in hand, namely:—

$$A = P\,R^{N}$$

as described in Chapter III. The derivation of the formula relating to the accumulation of an annuity or other periodic payment, namely:—

$$M = Ay\left(\frac{R^{N}-1}{r}\right)$$

is described in Chapter VI.

FORMULÆ AND SYMBOLS. The symbols adopted by the author differ somewhat from those given in the books on algebra, and in other mathematical works. They have, however, been chosen after much consideration in order to afford some indication of the factors they represent. The very full treatment which is given to the various formulæ is due to the fact that they are indispensable if a calculation has to be made at any rate per cent. not included in the published tables of compound interest. A detailed explanation of the symbols and formulæ is contained in Chapter X, dealing with the standard calculation forms prepared by the author.

LOGARITHMS. Throughout the book, the method of calculation is entirely by logarithms, since any attempt to arrive at the results by arithmetical methods would involve a serious waste of time, and a greater liability to errors in computation. The use of logarithms is fully explained in the usual arithmetical works, and also in the introduction to most of the published tables of logarithms, but a short chapter (No. II) has been included in order to make the book self-contained. There is not anything at all difficult in the use of a table of logs.: which is merely a very much neglected "ready-reckoner." There are several good tables giving seven-figure logarithms of the numbers from 1 to 108000.

MATHEMATICAL TABLES. There are many published tables of compound interest, which may be used to facilitate the various calculations, and which may be divided into three groups, namely:—

(1) *Tables giving the actual values* of £1, and of £1 per annum, for various periods at stated rates per cent. per annum. These tables are valuable in proportion to the number of rates per cent. for which the actual values are given. In using all such tables, a table of logs. is also required. In England, the one most generally used is known as Inwood's Tables (21st edition, 1880), and the new edition by Schooling (1899).*

(2) *Tables in which the actual values are not given, but which contain their logarithmic equivalents.* The tables of M. Fédor Thoman are of this type, and they are especially valuable, because they are worked out for many intermediate rates per cent. not given in Inwood's and other similar tables of compound interest, and also because they enable one to

* In America, Tables of Compound Interest, Discount, Sinking Funds, Annuities, etc., by Charles E. Sprague.

dispense with a table of logs., except as regards the actual sums of money involved in the calculation. They are particularly useful because all values are reduced to two factors only, namely, R^N, and a^n, by various combinations of which all the calculations may be made. The derivation and use of these tables are fully explained in Chapter IX.

(3) *Tables worked out on the "ready-reckoner" principle,* giving, for example, the sinking fund instalments, or the equal annual instalments of principal and interest combined, for £1, and multiples of £1, for various periods of years, at various rates per cent. Such tables may be very useful to some, but they have not any educational value whatever, and it is doubtful if they effect any actual saving of time when compared with the other types of tables, especially Thoman's, which are, when possible, always used by the author. The practical value of tables of this kind is limited by the number of rates per cent. actually worked out in detail, and the same applies to Inwood's and Thoman's tables.

If a problem be required to be worked out at any rate per cent. not given in any published table, such problem is impossible of solution by anyone not acquainted with the mathematical principles upon which all such tables are based. The object throughout has been to reduce these mathematical principles to the very simplest form, and to give such minute instructions, and to provide such standard forms of calculation, that anyone acquainted with the ordinary rules of arithmetic, and the use of a table of logs., may obtain the result required.

STANDARD CALCULATION FORMS. A special feature of the book is the series of standard forms, which have been specially prepared by the author, and by means of which all the calculations in the book have been made. They are fully described in Chapter X. The advantage of using these forms is that one or all of the three methods given on each form may be adopted; and it is generally advisable to make the calculation in two ways in order to prove the accuracy of the result and also to avoid any possible error due to a misprint in the mathematical table used. The three methods shown in each form are:—

A. by the mathematical formula.
B. by the published tables of compound interest.
C. by Thoman's Logarithmic Tables.

and in all cases the calculations are made by logarithms. The arithmetical method, based upon the published tables, is subject

to error, and is therefore unreliable. A supply of these forms is invaluable to anyone requiring to make many calculations of this nature, owing to their uniformity and also because they avoid any reference as to the particular method to be adopted. The formulæ, after a time, suggest the method. As a general rule the author uses, in the first instance, method C, by Thoman's Tables, as being the shorter, and also because these tables include a greater number of fractional rates per cent., than the ordinary published tables of compound interest. The factors being expressed in their log. values, a reference to the log. table is saved. The result is generally proved by logs. by method B, using the ordinary published tables of compound interest. In very few cases is it necessary to use method A, by formula, when the rate per cent. is worked out in Thoman's or other tables, but where the calculation is required at a rate per cent. not given in either table, method A, by formula, is the only one available. It is therefore necessary to become fully acquainted with the method by formula, and to use it to prove the result obtained by Thoman's method C, where it cannot be proved by method B, owing to the fact that the particular rate per cent. is not included in the table available. When it is required to ascertain the number of years with accuracy, the use of the formula is imperative, and the same applies to problems in which the rate per cent. is required. Very minute instructions as to the use of the forms are given in Chapter X, which contains also ten standard forms by which to ascertain the rate per cent. or the number of years.

Pro forma Accounts of Sinking Funds. Throughout the book the author has repeatedly laid great stress upon the supreme importance of following up the original calculation of the annual instalment by at once preparing a pro forma account showing year by year, how the fund should accumulate until maturity. To make these accounts fully answer their object they should be copied into a book kept solely for the purpose of preserving a permanent record of all such accounts, and not in the current ledger. This course will save endless trouble in future years. If any adjustment be made in the fund at any future time an amended pro forma account should be prepared and a reference made to the original account. If a copy of each calculation be forwarded to the Local Government Board it will materially assist the officials and simplify, if it does not entirely avoid, much subsequent correspondence between the Board and the local authority. Several pro forma accounts

have been prepared relating to examples given in the book, not only with regard to a normal sinking fund, but also as to an adjustment of the fund due to a variation in the period of repayment, the rate of accumulation and the income from investments. It is in such cases of adjustment that the provision of an account of this nature, showing the effect of the various changes until maturity, becomes particularly valuable.

THE REPAYMENT OF LOAN DEBT. Having in the earlier chapters described the methods of ascertaining the working formulæ and rules relating to the various classes of calculations, actual problems are next considered, beginning with the repayment of the loan debt of local authorities, taking as a basis the three alternative methods laid down in Sec. 234 of the Public Health Act, 1875. This section is a very concise statement of such methods of repayment, especially when supplemented by the non-accumulating sinking fund first mentioned in the model clauses inserted by the Local Government Board in provisional orders about the year 1893 and which have since been applied to many special Acts. The effects of the methods above mentioned are then fully discussed both as regards the lender and the rate or revenue account of the undertaking, illustrated by examples worked out in detail.

These three alternative methods apply equally to the repayment of the debt of privately owned commercial and financial undertakings, although the conditions in such cases are much more elastic and variable than is the case with local authorities. This is fully discussed in Chapter XIII.

PROBLEMS RELATING TO SINKING FUNDS. The remainder of the book is occupied by the discussion of actual problems relating to sinking funds proper, since the instalment and annuity methods do not involve the accumulation of any such fund but provide for the actual periodical repayment to the lender. Very few complications are likely to arise in the case of the instalment method, and any variations in the annuity method will follow the general rules as to a simple annuity. Such problems concern the amount in the fund at any time, the rate of accumulation of the fund, the rate of income to be received upon the present investments representing the fund, the period of repayment, or a combination of any or all of these factors.

The amount in the fund at any time may be the correct calculated amount which should stand to the credit of the fund, or may vary therefrom, resulting in a deficiency or a surplus.

A deficiency in the fund may be due to a fall in value, or a loss upon the realisation, of an investment representing the fund, but may also be caused by the accumulation of many minor past deficiencies in the annual income received from the investments; and, although it does not now often occur, may be due to a deficiency in the annual instalments set aside in past years. Cases have occurred, within the knowledge of the author, where the provision of a sinking fund in relation to an old loan has been entirely overlooked.

A surplus in the fund may arise in several ways; either by an increased rate of accumulation or by the payment into the fund of the proceeds of sale of part of the assets representing the security for the loan, or a realised profit upon the sale of an investment. In the case of commercial and financial undertakings, a surplus may arise upon the withdrawal of part of the loan from the operation of the fund. Two typical examples of this nature are very fully discussed in Chapter XVIII.

In all such problems it is first necessary to ascertain the actual position of the fund at the time the adjustment is required to be made, and this may be expressed in terms of the present investments and the future annual increment to accrue to the fund. The problem may be simplified by treating, as one factor, the "Annual Increment" of the fund which consists of the annual instalment and the income to be received from the present investments, whether the rate per cent. of such income is the same as the rate of accumulation or is different. Any variation in such rates may continue during the whole of the unexpired portion of the repayment period or for a portion of the period only. The term "Annual Increment" is fully discussed in Chapters XIV. and XXII.

The principal causes giving rise to a necessity to make an adjustment of the fund are variations in the rates per cent. of accumulation or of income upon the present investments and variations in the period of repayment, or a combination of both rate per cent. and period. Of the two causes a variation in either of the rates per cent. is the most probable for many obvious reasons, and it is very important that this should be carefully observed and immediately corrected, in order to avoid the necessity at some future time of having to make a substantial adjustment due to the accumulation of small errors. The longer a deficiency is allowed to accumulate the greater becomes the resulting burden imposed upon the correspondingly reduced number of the final years of the redemption period.

Calculation of a Typical Sinking Fund. In order to provide an example which may be used to illustrate the whole of the above problems, Chapter XV, Calculation (XV) 1, shows the method of ascertaining the annual instalment relating to a loan of £26,495 repayable in 25 years with an assumed rate of accumulation of $3\frac{1}{2}$ per cent. per annum. (Author's standard calculation form, No. 3 x.)

Methods of Adjustment. Throughout the book the fact that the particular method adopted is not the most direct one has been left entirely out of consideration, provided it has an educational value. In all cases, however, the shorter and more direct method has been shown and the results by the two methods compared. In the case of the adjustment due to a deficiency in the fund, in Chapter XV., four methods are given, which are summarised at the head of that chapter. The adjustment of a deficiency has been treated in this exhaustive manner, far beyond its relative importance, in order to present to the student a practical example illustrating the interdependence of the present value and future amount of £1 and of £1 per annum. In the above example, as well as in later ones, a statement has been prepared showing the various stages by which the amended annual instalment is ascertained, and this is followed in all cases by a further statement showing the final repayment of the loan by the operation of the sinking fund and the amended annual instalment rendered necessary by the variation in the original conditions.

Wherever required, the method has been reduced to a series of stages briefly stated giving a reference to the individual calculations. In the earlier parts of the book the actual details of the calculations are given in full or in the Appendix, but in later chapters only the final results are given owing to consideration of space and also because similar examples have previously been worked out.

The Annual Increment Methods. The adjustments next considered are those due to a variation in the rate per cent. either of accumulation or income from investments, a variation in the period of repayment, or a combination of the two factors of rate per cent. and period. As in the case of a deficiency or a surplus in the fund, the amended annual instalment is first ascertained by the deductive method fully described in Chapter XIX, which is based upon the consideration of the whole of the factors governing the fund. The same result is also shown by the annual increment (balance of loan) method

fully described in Chapter XXII. In the case, however, of a variation in the rate of accumulation accompanied by a variation in the rate of income from investments the latter factor is eliminated by merging it in the annual increment and dealing only with that annual sum. The varying rate of accumulation then becomes the only outstanding factor, and it is therefore possible to deduce a method which has been called "the annual increment (ratio) method," depending upon the ratio existing between the original and amended rates of accumulation. The whole of the calculations by the annual increment (ratio) method relating to a variation in the rate per cent. only, and also to a variation of the period of repayment only, bear a strong family likeness and are capable of being reduced to simple rules and formulæ, and this has been shown in detail. Having in Chapter XXVI discussed a combined variation in both factors of rate per cent. and period and having again deduced a formula therefrom the whole of the formulæ so obtained have been reduced to simple rules.

The Dates of Borrowing and Repayment. Up to this point all possible causes of the adjustment of a sinking fund have been exhausted, but the subject has been treated from the purely mathematical or actuarial standpoint, namely, that all loans are borrowed in one sum at the beginning of the financial year and that the annual instalments are set aside at the end of such year. The actual practical conditions are next dealt with, namely, that the loan is, as a rule, borrowed over a period of years, in various amounts and at various dates in any year, and is repayable sometimes over a period of years, but often on a given date. The subject is further complicated by the fact that varying periods are allowed for the repayment of loans sanctioned for different classes of outlay depending upon the life, or duration of continuing utility, of the individual works. And this varying period of repayment may be, and often is, complicated by practical variations in the dates of borrowing. This part of the subject has therefore been divided, by dealing first with loans authorised for outlay of one character only where the problem is not complicated by different periods of repayment due to the life of the asset. The problems relating to the dates of borrowing are sub-divided as follows:—

(*a*) Where the loan is borrowed over several years, in one sum in each year, and is repayable over a term of years in a prescribed period from the several dates of borrowing. Chapter XXVIII.

(*b*) Where the loan is borrowed over several years, in one sum in each year, and is repayable in one sum on a certain specified date. Chapter XXIX.

(*c*) Where the loan is borrowed in one or more years in varying amounts and at varying dates in each year and is repayable in one sum on a certain specified date, and it is further required that the revenue or rate account of each year of borrowing shall be charged with a proportionate part of the annual sinking fund instalment. Chapter XXX.

In the case of loans borrowed over a series of years, where the repayment is spread over a period equal to the extended years of borrowing, the amounts borrowed in each year may be treated as individual loans, and the only points to be considered are administrative, and relate to the number of sinking funds, namely, whether it is preferable to keep a separate sinking fund for each year's borrowings or to keep only one fund for the total loan. This is fully discussed in Chapter XXVIII, and, as there stated, cannot be applied to the redemption of stock.

In the case of loans borrowed over a period of years, raised by the issue of stock redeemable on a fixed date the several sinking fund instalments, although commencing at various dates, yet mature on the same date. The enquiry is still confined to loans in respect of outlay of one nature and having a uniform period of repayment. This class has been subdivided into two groups, and is fully discussed in Chapter XXIX.

1. Where the date of repayment is known at the time the money is borrowed.
2. Where the date of repayment is fixed after the sinking fund has been in operation for a number of years, and an adjustment becomes necessary.

The apportionment of a part of a full year's instalment to be charged against the revenue or rate account of the year in which the money is borrowed is treated fully in Chapter XXX. As a rule, this may be, and generally is, ignored; but the particular circumstances in connection with a large loan may render it advisable to make a charge of this nature. There are several interesting features in the method which is illustrated by the example in Chapter XXX, and which may be compared with the instalments to be set aside when the year of borrowing is not charged with any such proportional annual instalment, as in Chapter XXIX. Stated briefly, the effect is to ante-date

the charge to revenue or rate and to impose an increased burden upon the years of borrowing. The third year of the sinking fund period is charged with the same amount under each method because the repayment period is assumed to commence at the conclusion of the first year of borrowing. Such increased annual burden during the earlier years operates by way of relief to the remainder of the repayment period, but only to a slight extent. This problem furnishes another example of an adjustment being required in consequence of irregular contributions to the fund during the earlier years.

THE LIFE OF THE ASSET, AND THE EQUATION OF THE PERIOD OF REPAYMENT. Having dealt with problems relating solely to the adjustment of the sinking fund, owing to causes of a purely actuarial or mathematical nature, there is still to be considered the more difficult subject of the variation in the periods allowed for the redemption of loans for large public works, where each component part of the outlay has a different life or duration of continuing utility. This variation in the redemption period has not any disturbing effect when the loan is authorised for one class of outlay only, or is in respect of several classes of outlay, each having the same period of repayment. But loans are now often authorised for large public works which include various classes of outlay, each class having its own redemption period, based upon its duration of continuing utility, and also forming a variable proportion of the total cost, and it is required that the total loan shall be repaid on the same date. In such cases it becomes necessary to ascertain the equated period of repayment. The same necessity arises on the consolidation of existing loans repayable at various future dates, but in such cases the problem is further complicated by the amounts then standing to the credit of the individual sinking funds, the value of the investments representing each fund, the rate of income arising therefrom, and also the incidence of the present redemption charges upon different departments of the local authority. It may be stated generally that the problem of the equation of the period of repayment applies equally to all such cases and that in fixing the equated date of repayment there are two interests to be considered, namely, the loanholder, who looks only for the due payment of his principal, and the annual interest thereon, and the individual ratepayer who is required to provide his proper portion of the annual amount which Parliament, or the government department concerned, has laid down in principle as the annual wastage of the assets created

out of the loan. This annual wastage of the asset is imported into the problem owing to the fact that the period allowed for the repayment of the loan is based upon the life, or duration of continuing utility, of the asset. As regards the loanholder, the problem is a simple one. In all cases he will be repaid, at some future date, the amount which he originally advanced to the local authority, and, in addition, he will receive until the repayment of the loan, interest at the rate per cent. originally fixed. The only question remaining therefore, so far as he is concerned, is the relation between the rate of interest agreed to be paid by the local authority and the rate per cent. obtainable upon the open market when the loan is proposed to be repaid. This arises only upon the consolidation of existing loans repayable at fixed future dates where the effect of consolidation is to vary, and generally to anticipate, the date at which the loan was originally repayable. As regards the loanholder, therefore, the most important factor is the period during which he will continue to receive interest upon the loan at the present rate payable by the local authority. If the present rate so payable be a high one and the current rate, to be obtained upon the open market, at the time when the local authority propose to repay the loan, be expected to be lower, the loanholder will naturally object to any variation of the original conditions, and, *per contra,* he will gladly accept an earlier repayment of the loan if thereby he may expect to obtain a higher rate of interest upon his investment. Consequently the loanholder must be consulted, and his consent obtained, before any change be made in the original conditions upon the consolidation of loans. This uncertainty as to the future rate of interest is one of the reasons why Parliament has, for some years past, refused to sanction the issue of an irredeemable stock in consequence of the difficulty in applying the amount in the sinking fund to its proper purpose. The stock can thus only be redeemed by purchase upon the open market, and the premium paid upon such occasions cannot be taken out of the sinking fund, but must be charged against the revenue or rate account of the current year.

The ratepayer, on the contrary, is in a very different position, in that the money paid to the loanholder by way of interest upon the loan, and the annual sums set aside out of revenue or rate to redeem the debt are paid by him. But the ratepayer comes and goes, whilst the loanholder goes on for ever,. or at least until his loan is repaid. The loanholder naturally considers the value of his investment and the interest

to be derived therefrom, and the state of the money market both at the present time and in the future are to him very important factors. The ratepayer, on the contrary, considers only the annual amount paid by him by way of rate, and compound interest is to him a negligible, if not an unknown term. In addition, he is never consulted individually as to the annual amount of rate which he may be called upon to pay. He may be invited to attend a meeting called to approve or disapprove of a Bill to be laid before Parliament to authorise the spending of money on capital account, but he is generally ignorant of the matter, and is too busy trying to earn the amount he has to pay by way of rate, to attend any such meetings. The result is that the final adjustment is left entirely to the officials of the local authority subject only to the control of Parliament or the Local Government Board, and the next step therefore is to investigate the methods generally adopted in order to arrive at the equated period of repayment and the consequent amended annual sinking fund instalment to be charged to revenue or rate.

Before doing so, however, it should be pointed out that any necessity to fix the equated period did not arise to any great extent until it became the common practice of local authorities to issue stock or to consolidate existing loans repayable at various dates. Prior to that time, any variation in the periods of repayment allowed for different classes of outlay was met by keeping separate funds for each amount of loan having the same repayment period and allowing each fund to mature at the due date. The relation between the life of the asset and the consequent annual loan charge upon the revenue or rate accounts of successive years is fully discussed in Chapter XXXII, where it is found that the variation in the periods of repayment allowed is not of itself a cause of an equation being required, which depends upon a combination of two factors, namely, the variable period of repayment and the obligation to repay various loans on one instead of on different dates. The problem arising on the consolidation of stocks or loans repayable at various dates is exactly similar in principle although arising in a somewhat different manner, but is further complicated by the amount in the fund at the time of making the adjustment.

THE EQUATION OF THE PERIOD OF REPAYMENT. The equation of the period of repayment has been considered from two points of view, namely, one relating to the method of

ascertaining the equated date and the other to the incidence of the annual burden upon the revenue or rate account. These two points are fully treated in Chapters XXXII, XXXIII and XXXIV. The method generally adopted to find the equated period is the arithmetical one known as the "equation of payments," which is fully described in Chapter XXXII. It is there proved, by two examples worked out in detail, that the equated period as generally adopted is not the true equated period and that the effect of adopting it is to extend the period of repayment beyond the true or mathematically equated period. This may not be important in many cases, but may be extremely so in the case of very large loans; and, if it be necessary to make such an adjustment at all, it is surely imperative that the principle upon which it is made is scientifically accurate.

Having described the proper method of finding the true equated period, Chapter XXXIII is occupied with an examination, illustrated by the actual example used in Chapter XXXII, of the effect of the generally adopted practice of fixing the amended annual sinking fund instalment by spreading the burden equally over the whole of the equated period as if it related to an original loan, repayable in such period, the whole of the loan representing outlay of one character only, having a life or period of utility of that length. The method is a simple one, but is wrong in principle although it has received the approval of many years' adoption.

If the preliminary stages in the sanction of a loan be carefully reviewed it will be recognised that much thought and care are expended in determining the proper periods to be allowed for the repayment of loans authorised for different classes of outlay. The whole question is still in a transition state, and great divergence exists in the conditions now in force; the only recognised factors being that in future the annual charges for redemption of the debt shall bear a definite relation to the life of the asset, with a further extension of the principle, that even in the case of works of almost permanent utility the repayment shall not extend beyond a certain number of years. This latter requirement is to protect the interests of future generations of ratepayers. The relation between the period of repayment and the life or duration of continuing utility of the asset is imposed in order to ensure that the present generation shall contribute, year by year, the proper portion of the wastage of the asset. In the case of a loan raised for works comprising outlays of varying nature with varying periods of repayment and where separate sinking funds are kept in respect of each class of outlay the

principle is carried out exactly because the earlier years bear the heaviest burden, as they should do, owing to the fact that classes of outlay having a short life will be worn out and require replacing at the end of the period. Under these conditions the loan is entirely repaid by the time the works cease to be of utility or are worn out.

THE EQUATION OF THE PERIOD OF REPAYMENT, AND THE INCIDENCE OF TAXATION. THE ANNUAL INSTALMENT. Under the present practice on equation the above principle is departed from, and the burden is spread equally over the equated period with a total disregard to any period of utility, and as demonstrated in Chapter XXXIII, there is actually considerable relief to the early years of the equated period as well as an entire removal of any burden during the years of the original period beyond the equated period. As a consequence, the whole of this relief is imposed as an additional burden of considerable magnitude upon the final years of the equated period. There is here a total reversal of the generally accepted principle of spreading the repayment of the loan over the period represented by the life of the asset, accompanied by an absolute injustice to a section of the ratepayers. By adopting the equated method in general use, as applied to works consisting of various classes of outlay, and also on the consolidation of loans, the present generation relieve themselves of a liability to contribute their fair share of the burden which has been fixed after careful enquiry by Parliament; and thereby impose an extra burden upon future years. In addition they also postpone the repayment of the loans with shorter periods which would have been repaid during the earlier years, and the result is that money cannot properly be reborrowed to replace assets with a shorter life than the equated period because, when they are worn out, the original loan has not been repaid by means of the sinking fund. A remedy for this state of affairs, so far as the annual instalment only is concerned, is pointed out in Chapter XXXIV, namely, by spreading the burden over the equated period, not by an equal annual instalment, as is the present practice, but by instalments of varying amounts approximating to those originally imposed which were based upon the life of the asset. The principle of this method is to ascertain, first, the amount of loan which will be provided at the end of the equated period, by the accumulation of the annual instalments as originally fixed. The amount of such instalments at the end of the number of years for which they would have been set aside under

the original conditions should be ascertained, and if any of these periods are shorter than the equated period, the amount at the end of such periods should be further accumulated until the end of the equated period. The difference between the amount of loan so ascertained and the total amount of the loan ultimately repayable will represent the amount to be provided by supplementary annual instalments to be spread over the equated period, due to the fact that the equated period is shorter than the original periods allowed for the repayment of the parts of the loan with longer periods, and that the relief afforded by the equation to these later years should be borne equitably by each year of the equated period. Strictly speaking, such supplementary annual instalments should be graded in some manner proportionate to the original annual instalments which were based upon the life of the asset, and although the calculation is fully described it is somewhat intricate, and the justice of the case will generally be met by spreading this supplementary annual instalment equally over the equated period.

The Equation of the Period of Repayment, and the Incidence of Taxation. Interest upon the Loan. The result of spreading the annual instalment over the equated period in proportion to the instalments before equation is shown in Table XXXIV, J, where the original annual instalments are corrected in this manner. On referring to Table XXXIII, B, it will be seen that the effect of equating the period is to throw a heavy additional burden upon the final years of the equated period in respect of interest upon the loan. In Chapter XXXIV, a method is described of distributing the redemption charge (the annual instalment) equitably over the equated period, and in Chapter XXXV a similar course is adopted with regard to the interest upon the loan, with the result shown in Table XXXV, C. By combining the correctly equated annual instalments shown in Table XXXIV, J, with the correctly equated annual interest charges shown in Table XXXV, C, the total annual loan charges during the equated period may be ascertained as shown in Table XXXV, F. The subject is so important that the result has been shown in graphic form, which is fully described and explained in Chapter XXXV. In order to express in actual values the effect of the above adjustment both as to the instalment and interest upon the loan it may be stated in terms of annual rate. It has been given in evidence before a Parliamentary Committee that in one

particular case of consolidation of loans the immediate effect of an equation of the period of repayment was a saving of threepence in the £ in the annual rate. In this connection it should be remembered that as shown in Table XXXIII, C, there is not any difference in the annual charges for interest upon the loan during the early years of the equated period, before and after equation, but that the decrease is entirely in the annual instalment. Adopting the figure of threepence in the pound as a standard, the result in the present case would be as follows, including the interest upon the loan as well as the annual instalment. The following figures may be converted into American currency by adopting the equivalent of $2\frac{1}{2}$ cents to the dollar instead of 6d. in the pound:—

Original redemption period.			Decreased rate per £ of Annual Value.	Increased rate per £ of Annual Value.
Equated period ...	5 years	...	6·52 pence	...
	10 years	...	3·20 pence	...
	8 years	...	...	11·85 pence.
Post-equated period ...	6 years	...	9·76 pence	...
	16 years	...	2·79 pence	...

Actual Calculations. The method adopted throughout the book is to insist upon a very careful scrutiny of the present and future conditions and also of the actuarial and mathematical principles involved. It is very important to prove all ascertained results, both as to method and accuracy of computation, seeing that the actual working out of the fund will occupy many years and the effect of any present error will be serious if it be allowed to accumulate for any length of time. Mere repetition of the actual calculation is not sufficient. A far preferable method is to work out the operation of the fund year by year by the arithmetical method as shown in the pro forma accounts already referred to. This should be done in all cases without exception before the problem is finally disposed of, but the method is laborious and much time is wasted if the original calculation be wrong. The best way is to prove the result by mathematical means which are much shorter, either by adopting an alternative method, of which many instances are given in the various chapters, or by comparing the amended with the original annual instalment and accounting for the difference. The actual arithmetical calculation of the pro forma sinking fund accounts may be left to a junior official, but it will save him considerable time if the senior first ascer-

tains the amount in the fund at the end of every five or ten years by means of the tables or otherwise as described in the various chapters. The whole of the calculations in the book have been verified in this manner and in many cases the method of proof is shown in detail. In cases, however, where it is not shown the verification has been made and is only omitted for want of space.

Section I.

Mathematical Principles

CHAPTER II.

LOGARITHMS.

ADVANTAGE OF USE OF LOGS. HISTORY. CONNECTION BETWEEN LOGS. AND ARITHMETICAL AND GEOMETRICAL PROGRESSIONS. DEFINITION. VARIOUS ARITHMETICAL CALCULATIONS BY LOGS. LOGS. OF NUMBERS BETWEEN EVEN MULTIPLES OF 10. CHARACTERISTIC. MANTISSA. METHOD OF DIVIDING A LOG. WITH A NEGATIVE CHARACTERISTIC. METHOD OF DIVIDING ONE LOG. BY ANOTHER.

In a work of this nature, dealing with calculations which are based upon the higher branches of mathematics, it is obvious that the ordinary methods of arithmetic are inadequate, and that the aid of logarithms must be invoked even if the fullest use be made of the various published tables of compound interest. There is a limit to the number of rates per cent. which may be included in any table, and it is often required to make a calculation at a rate per cent. not worked out. In such cases it is necessary to revert to the original formulæ, all of which involve raising numbers, containing as many as five or six figures, to the power of the number of years, and the method of continued multiplication becomes too laborious and uncertain. Even when using the tables it is always necessary to multiply or divide by the numbers (containing five or six figures) given in the tables, and a great saving of time and labour is effected by doing this by the aid of logarithms. But beyond the little time expended in becoming familiar with the method of using such a table, there is not any greater difficulty than in using any ordinary commercial ready reckoner.

This chapter deals only generally with the subject of logarithms, and as the use of this book cannot be complete without a copy of Inwood's or other similar tables, and a reliable table of logs., for a fuller acquaintance, reference must be made to the introductory chapter which will be found in most log. tables, or else to some good advanced arithmetic.

Logarithms were invented by John Napier, Baron of Merchiston, in Scotland, who published his first work in Edinburgh in 1614. This work contained only the logarithms of natural sines, and are not what are now known as Naperian or hyperbolic logarithms, which are used in mathematical investigations only and are not the logarithms in common use

to-day. Napier died in 1617, and a further work by him, edited by his son, was published in 1619.

The first published table of decimal or common logarithms was published by Henry Briggs, Professor of Geometry at Gresham College, London, and afterwards Savilian Professor of Geometry at Oxford, who visited Napier in 1615. Briggs published his table in 1617 (after the death of Napier), and these logarithms which are in common use to-day are calculated to a base of 10. Briggs' first tables contained only the logarithms of numbers from unity to 1,000 to 14 places of decimals. The arithmetical calculation of logarithms as used by Briggs, is a very laborious process, and it was not until 1628 that the table was extended for numbers from unity to 101,000 by Briggs and Adrian Vlacq, of Gouda, in Holland. But the arithmetical method of Briggs was afterwards superseded by shorter methods depending upon more advanced mathematical rules, and there are now, as the result of all this labour, very accurate tables which are in universal use, and by means of which very intricate calculations may be made by very simple methods.

The principle upon which a logarithm is based is exceedingly simple, and is founded upon the relation existing between an arithmetical and a geometrical progression. An arithmetical progression is a series of numbers each of which is found by adding a constant number to the previous term in the series, the constant number so added being called the ratio. A geometrical progression is a series of numbers each of which is found by multiplying the previous term in the series by a constant number, such constant multiplier being called the ratio. Taking a series of numbers in geometrical progression, with a ratio of 10, which is the one adopted in the Briggean or decimal or common logarithms, and commencing the series with unity, the following series is obtained:—

Geometrical Progression,	1.	10.	100.	1,000.	10,000.	100,000.	1,000,000.

Taking another series of numbers in arithmetical progression, with a ratio of 1, and commencing the series with 0, the following series is obtained:—

Arithmetical Progression,	0.	1.	2.	3.	4.	5.	6.

The above geometrical progression will now be re-written expressing each term by the power of 10 which it represents, and under it the above arithmetical progression, as follows:—

Geometrical Progression,	10^0	10^1	10^2	10^3	10^4	10^5	10^6
Arithmetical Progression,	0	1	2	3	4	5	6

It will be at once noticed that the index of each term in the geometrical progression is the same as the corresponding term in the arithmetical progression.

If it be assumed that the terms in the arithmetical series are the logarithms of the corresponding terms in the first geometrical series, this is exactly what Briggs did when he adopted 10 as the basis of his system, as follows:—

The log. of	1.	=	10^0	=	0
	10.	=	10^1	=	1
	100.	=	10^2	=	2
	1,000.	=	10^3	=	3
	10,000.	=	10^4	=	4
	100,000.	=	10^5	=	5
	1,000,000.	=	10^6	=	6, and so on,

and therefrom the definition of a common logarithm may be expressed, viz., *the logarithm of a number (to the base 10) is the power or index to which 10 has to be raised to produce that number,* for example: the logarithm of 10,000, to the base 10, is the power or index 4, to which 10 has to be raised to produce 10,000, but as 10 is the common base to which all numbers are reduced, the indices, or logarithms, only are required, and the decimal part of this index is all that is given in the log. tables.

The logarithms of numbers which are even powers of 10 may be ascertained in the above simple manner, and attention has been drawn to the enormous labour involved in calculating the logarithms of the intermediate numbers. It is not necessary to enquire deeper into the methods, but only to accept the tables which are the product of that labour. The logs. which have been already found may be used to illustrate the various advantages of their use, taking familiar arithmetical calculations, using the actual numbers in the ordinary way, and then repeating the calculations, using the logs. of the numbers instead of the actual numbers, as follows:—

Multiplication:

$10 \times 1{,}000 \quad = \quad 10{,}000.$ by logs., $1+3=4$, or log. 10,000

Division:

$100{,}000 \div 100 = \quad 1{,}000.$ by logs., $5-2=3$, or log. 1,000

Involution:

$100^3 = 1{,}000{,}000$. by logs., $2 \times 3 = 6$, or log. 1,000,000

Evolution:

$\sqrt[3]{1{,}000{,}000} = 100$. by logs., $6 \div 3 = 2$, or log. 100

It will be seen therefore that by the use of logs.—

$$\left.\begin{array}{l}\text{Multiplication}\\ \text{Division}\\ \text{Involution}\\ \text{Evolution}\end{array}\right\}\begin{array}{c}\text{of ordinary}\\ \text{numbers}\end{array}\Bigg\}\text{ becomes }\left\{\begin{array}{c}\text{addition}\\ \text{subtraction}\\ \text{multiplication}\\ \text{division}\end{array}\right\}\begin{array}{c}\text{of the}\\ \text{respective}\\ \text{logs.}\end{array}$$

It is a common practice when multiplying together two powers of 10, such as $(10 \times 1{,}000)$ to write down 1, and add 4 cyphers, thus, 10,000, being the sum of the cyphers in the two numbers. This is actually a logarithmic method of calculation often used by people who do not know anything about logarithms. This is the principle of the algebraical theory of indices, of which the following is an illustration:—

Multiplication: $x \times x^3 = x^{(1+3)} = x^4$

Division: $x^5 \div x^2 = x^{(5-2)} = x^3$

Involution: $(x^2)^3 = x^{(2\times 3)} = x^6$

Evolution: $\sqrt[3]{x^6} = x^{(6\div 3)} = x^2$

The above operations correspond to the previous illustrations in which the actual powers of 10 were used. In the algebraical form above, 10 has been replaced by x, with the result that similar logs. are obtained in each set of examples.

Up to this point only whole numbers have been considered which are even multiples of 10, and of which the logs. are whole numbers above unity, and it has been ascertained that the logarithm of 10 is 1, that of 100 is 2, and so on for any even power of 10. It is therefore obvious that the logs. of numbers less than 10 must be fractions. This also applies to numbers between 10 and 100, the logs. of which must be between 1 and 2, and equally to numbers between any two consecutive powers of 10, which logs. consist of a whole number and a fractional part, the whole number being the log. of the next lower power of 10.

A logarithm then consists of two parts—the integral part, which is called the *Characteristic*, and the fractional or decimal part, which is called the *Mantissa*, and all logs. are expressed in decimals, usually to 7 places.

The Mantissa (or fractional part) is always positive, and is

always the same for any one combination of figures, irrespective of the place of the decimal point.

The Characteristic represents merely the position of the decimal point in the number which the log. represents, and changes only after passing each power of 10. The characteristic is in all cases the power to which 10 has to be raised to produce the next lower power of 10.

The logarithm tables contain only the mantissa part of the logarithm corresponding to the particular combination of figures forming the number of which the log. is required, and since these figures may, according to the position of the decimal point, represent either whole numbers or fractions, the characteristic is positive in the case of a whole number, and negative in the case of a fractional number.

It will be noticed, on referring to the logs. of the powers of 10, referred to above, that the log. of 1,000,000 is 6, or one less than the number of integral figures (seven) in the number, and similarly with the other powers of 10, and the rule applies generally, as will be seen by taking the logarithm of the number 26495.

In the table of logs. opposite 26495 are the figures 423,1639 which is the mantissa of the log. of any number containing the figures 26495 in this order, whether preceded or followed by any number of cyphers. The actual position of the decimal point determines the characteristic or integral part of the log. as follows:—

$$\begin{aligned}
\text{Log. } 26495 &= 4{\cdot}423\ 1639\\
2649{\cdot}5 &= 3{\cdot}423\ 1639\\
264{\cdot}95 &= 2{\cdot}423\ 1639\\
26{\cdot}495 &= 1{\cdot}423\ 1639\\
\hline
2{\cdot}6495 &= 0{\cdot}423\ 1639\\
\hline
{\cdot}26495 &= \bar{1}{\cdot}423\ 1639\\
{\cdot}026495 &= \bar{2}{\cdot}423\ 1639\\
{\cdot}0026495 &= \bar{3}{\cdot}423\ 1639, \text{ and so on.}
\end{aligned}$$

On comparing the above logs. with the logs. of the powers of 10 previously given, it will be noticed, for instance, that 264.95 (being above 100 and below 1,000) has the characteristic

2 as previously explained, and it will be further noticed that as the decimal point in the original number is moved place by place to the left (equivalent to dividing the previous number by 10) the characteristic of the logarithm is reduced by 1. But in the case of 2·6495 the characteristic becomes 0, and as the number is further divided by 10 and the decimal point moved still further to the left, it becomes -1, -2, -3, and so on. The characteristic being the only negative part of the log., the minus sign is placed over it instead of to the left.

A glance at the above logs. will show that the characteristic follows two rules, viz.:—

(1) *In the case of numbers greater than unity, the characteristic is one less than the number of integral figures in the number, and is always positive; and*

(2) *In the case of numbers less than unity, the characteristic is one more than the number of cyphers after the decimal point in the number, or is the same number as the place from the decimal point which the first significant figure occupies; and is always negative.*

The usual published tables of common logarithms give the mantissa for each number from unity to 108,000, and the logs. of all numbers containing 5 figures may be found at one reference. If the number of which the log. is required contains more than 5 figures, the corrected log. is found by reference to one of the tables of proportional parts given in the margin of the tables, but all the published tables describe so fully how this is done that it is not necessary to repeat it here.

There are also several other practical operations required which are fully explained in the tables, amongst others, (1) finding the antilog. or the number corresponding to any logarithm, and (2) the method of dealing with logs. having negative characteristics, either by addition or subtraction, which follows the ordinary rules of algebra.

Special attention should, however, be given to the rules as to multiplying or dividing a log. with a negative characteristic. The following method of dividing such a log. is used by the author in order to find the value of the factor R, and differs from the method given in the tables, but is simpler. It is as follows:—

Having obtained the log. of R^N ($N=20$ years), viz., it is required to divide the log. by 20, in order to obtain log R,	$\bar{2}\cdot987\ 8003$
proceed by adding 20,	$20\cdot$
	$=18\cdot987\ 8003$
Divide this log. by 20 =	$0\cdot949\ 3900$
and deduct 1, to correct the addition of 20, divided by 20, =	$1\cdot$
Leaving the required log.	$\bar{1}\cdot949\ 3900$

It is sometimes required to divide one log. by another, as in Calculation XXXII, E., in order to find the number of years, N, in an equated period at a given rate per cent., knowing the value of the factors R^N and R. If both the logs. are positive or negative, they may be treated as ordinary numbers and the corresponding logs. found in the usual way, but if their characteristics are, one plus and the other minus in sign, they must be reduced to the same sign.

CHAPTER III.

SIMPLE AND COMPOUND INTEREST.

SIMPLE INTEREST. AN ARITHMETICAL PROGRESSION. FORMULÆ. TABLES. INCIDENTAL USE OF THE TABLES.

COMPOUND INTEREST. A GEOMETRICAL PROGRESSION. DERIVATION OF THE FORMULA, $A=P\,R^N$, RELATING TO COMPOUND INTEREST, FROM THE ALGEBRAICAL FORMULA, $l=a\,r^{(n-1)}$, RELATING TO A GEOMETRICAL PROGRESSION. EXPLANATION OF TERMS. DIFFERENCE BETWEEN THE AMOUNTS OF £1 AND OF £1 PER ANNUM AT THE END OF 1 YEAR. "PRESENT VALUE" COMPARED WITH "PRACTICAL DISCOUNT."

SIMPLE INTEREST, AN ARITHMETICAL PROGRESSION. Simple interest is an arithmetical progression, and the amount of any sum of money, at the end of any given term, may be ascertained by continued addition of the interest upon the sum for one year, or other period, at the stated rate per cent. It is the method in general use in all commercial and financial transactions, although in cases where balances in an account current are struck at stated periods, it may partake of the nature of compound interest. The main feature of this method is that the calculations may relate to varying sums, varying times, and varying rates per cent., and are expressed by the formula :—

$$\text{Interest,} = \frac{\text{Principal} \times \text{rate per cent. per annum} \times \text{years}}{100}$$

and the ascertained amount of interest is stated in the same terms as the principal, whether pounds sterling, shillings sterling, dollars or other currency. All such calculations are extremely simple, and many tables are published giving the amounts of interest on varying amounts of principal for varying periods, whether days or years. The above formula is the one used to calculate the amount of interest for one or more years. If it be required to calculate the amount of interest for any number of days at a given rate per cent. per annum, the formula becomes :—

$$\text{Interest,} = \frac{\text{Principal} \times \text{rate per cent. per annum} \times \text{number of days}}{100 \times 365}$$

The utility of any table of simple interest is limited only by its size, and it is very easy by means of the above formula to ascertain any required sum not given in the table. There are several modifications of this method to suit individual or special requirements which do not, however, require special mention.

Whilst on the question of simple interest, there is an interesting method of using such tables which may not be generally known. It is often required to ascertain the amount of rent, or other annual sum for a given number of days. If, for instance, it is required to ascertain the amount of 97 days' rent at £865 per annum, proceed as follows:—Multiply the annual rent £865, by 20=£17,300; refer to the tables and ascertain 97 days' interest upon £17,300 at 5 per cent. This will be the amount of 97 days' rent.

Similarly, the annual rent may be multiplied by 25 and interest upon the product ascertained at 4 per cent., but the above method is the simplest as it involves multiplying by 2 only. As a matter of fact any other equivalent multiplier and rate per cent., having 100 for their product, may be used. This method may be applied to ascertain the proportion of the annual sinking fund instalment to be set aside in respect of a loan borrowed at various dates in one year as afterwards pointed out in Chapter XXX.

Compound Interest, a Geometrical Progression. Compound interest differs from simple interest in that it is a geometrical progression in which the rate per cent. is always uniform during the whole period, and the periods are all equal, whether years, half years, months, or otherwise. There are several published tables of compound interest, and many tables have been calculated for special purposes. The one most generally used in England is by William Inwood (18th Edition published 1880), commonly referred to as "Inwood's Tables." A new and much improved edition was issued in 1899, revised and extended by Mr. William Schooling.

Tables of this character are extremely useful, and provide for the majority of calculations required to be made by Local Government and municipal authorities, actuaries, accountants, bankers and valuers, and the officials of commercial and financial undertakings.

Derivation of the Formulæ. It is a very interesting study to analyse the tables mathematically and to derive each

table from the simple algebraical formula used to find the last of a series of numbers in a geometrical progression, viz.:—

$$l = ar^{n-1}$$

where a = the first term,
l = the last term,
r = the constant factor or ratio,
n = the number of terms in the progression.

A geometrical progression consists of a series of numbers which increase or decrease by a constant factor or common ratio, and many problems may be solved by means of the algebraical formulæ relating to such a progression, namely, the sum of a series, either finite or to infinity, the insertion of a number of geometric means between two numbers, and finding the last term of a series. Problems involving compound interest, however, include only the first term, the ratio, and the last term, all of which may be determined by means of the algebraical formula with only slight modification. The factors (which remain unchanged except as regards the actual symbol) are as follows:—

a = *the first term of the progression*, which corresponds to the principal sum (P) at the beginning of the number of years.

l = *the last term of the progression*, which corresponds to the amount (A) of the principal sum (P) at the end of the number of years.

r = *the common ratio*, or the number by which each term in the progression is multiplied in order to find the succeeding term. In the formulæ relating to compound interest this is expressed by the symbol (R) because when dealing with annuities, a symbol is required to represent a new factor $(R-1)$ which is denoted by (r), and which will be explained later.

n = *the number of terms in the progression*, and is the only factor in the algebraical formula requiring any alteration in the sense in which it is used. In both formulæ the ratio acts in exactly the same manner, or once during each interval in the progression, and it acts upon each term except the last. In any progression, the number of intervals between the terms is one less than the number of terms, or, as it may be expressed:

$$(n) \text{ intervals} = (n-1) \text{ terms.}$$

In the case of compound interest, the intervals are years, or other equal periods of time, consequently the algebraical formula is altered by substituting (N) years for $(n-1)$ terms, using the capital (N) to denote the number of years in order to distinguish it from the small (n) which denotes the number of terms in the algebraical formula.

Substituting the amended symbols as above,

$l = ar^{n-1}$ becomes, $A = P\,R^N$,

and the above symbols have the following meaning throughout the book:—

A = *the amount*, or the ultimate sum to which the present sum (P) will accumulate in (N) years at the ratio or constant factor (R). This symbol will be used to denote this factor whether it represents the ultimate sum required to be found at the end of the period; or the given sum due at the end of the period, of which it is required to find the present value (P).

The use of the word "amount" is different from the usual meaning attached to it in ordinary language, and it is very necessary to distinguish it from a sum of money. A very much better word would be "accumulate."

P = *the principal or present value*, and denotes a sum of money in hand, or due, now. It also denotes:

(1) the present value of a definite sum of money (A) due at the end of a stated period of years, and

(2) the present value of an annuity or other periodic sum (Ay) payable or receivable at the end of each of a stated number of years or periods.

These two factors (A) and (P) are intimately related. (P) is the first term, and (A) the last term, of a geometrical progression. (P) is the present value of (A) due at the end of a stated term, and (A) is the amount to which (P) will accumulate during that period.

R = *the ratio or common factor*, and denotes the rate of increase (expressed in terms of unity) in each term of the progression. It does not denote the rate per cent. per annum, although it is derived directly from the rate per cent. It is, in all cases, £1 increased by interest upon

£1 for one year at the rate per cent. in question; in the case of 5 per cent. it is 1·05, as will be clearly shown in Calculation (IV) 1, and so on for every other rate per cent. The ratios corresponding to each rate per cent. from $\frac{1}{4}$ to 7 per cent. are given later in Table No. V. A, together with the corresponding logarithms. In calculations involving compound interest the actual rate per cent. is never used, but only in its relation to £1 by way either of a ratio (R) or of the interest upon £1 for one year (r).

N = *the number of years,* or other equal periods, and, as already explained, must not be confounded with (n) in the algebraical formula for a geometrical progression which denotes the number of terms in the progression. This number of terms includes the first term, but in the case of compound interest the number of years is one less than the number of terms in the progression; therefore (N) years $= (n-1)$ terms. In the case of annuities, a modification of the above formula will be required, the derivation of which from the formula ($A = P\ R^N$) will be fully explained. This modified formula will contain additional symbols, namely—

Ay = *the annuity or other periodic sum,* to be set aside, paid or received at the end of each year or period.

M = *the amount of the annuity* or other periodic sum (Ay) accumulated for a given number of years or periods (N) at a given rate per cent. This symbol bears the same relation to a periodic sum (Ay) as (A) bears to a present sum (P).

r = *the interest upon £1 for one year* or period at the stated rate per cent. It is found from the above ratio or common factor by deducting unity therefrom. The values of this factor for the various rates per cent., and the corresponding logs., are shown in Table No. V. A, which will be given later.

The above formula, $A = P\ R^N$, with its various modifications, may be used to find factors which are sufficient to solve all questions of compound interest in relation to sinking funds and annuities. The actual values of each factor are capable of being tabulated for varying terms at varying rates per cent.; and to make them generally useful the results are stated in the

published tables in terms of £1 so that any problem as to other amounts may be solved by multiplying or dividing the actual figure in the problem by the amounts given in the published tables. In the old edition of Inwood these tables, I. to V., are given separately; but in the new edition, Tables I. to IV. are shown in four separate columns in one table. Throughout the book they will be referred to as Tables I. to V., and anyone using the new edition will refer to the corresponding column in the table on pages 50 to 85.

THE DIFFERENCE BETWEEN THE AMOUNTS OF £1 AND OF £1 PER ANNUM AT THE END OF ONE YEAR. It is important to remember that in all calculations involving (P) the sum of money which it represents is due or in hand at the beginning of the first year of the period. In the case of annuities, the annual sum is assumed to be set aside, paid, or received at the end of the first and every subsequent year of the period. This is very important, sufficiently so to justify the following extracts from the tables:—

Table I.	The amount (A) of (P) £1 at the end of one year at 5 per cent. is	£1·05
Table II.	The present value (P) of (A) £1 due at the end of one year at 5 per cent. is	£0·9524
Table III.	The amount (M) of (A*y*) £1 per annum at the end of one year at 5 per cent. is	£1·00
Table IV.	The present value (P) of (A*y*) £1 per annum for one year at 5 per cent. is	£0·9524

From the above it will be seen that the amount of £1 at the end of one year (£1·05) is greater than the amount of £1 per annum at the end of one year (£1) because the £1 is in hand and bears interest during the first year, whereas the annuity of £1 per annum is not due until the end of the year. But on comparing the present value of £1 due at the end of one year, and the present value of £1 per annum due at the end of one year, they are the same (viz., £0·9524) because they are both due at the same time.

Problems may arise involving a variation from this principle when dealing with purchases on the deferred payment system. In such cases, the annual instalment of principal and interest combined is generally payable at the end of the first and

subsequent years, in the above manner, but it sometimes happens that the agreement provides that the first payment shall be made at the beginning of the first year which makes an important alteration in the method. Such problems, however, rarely arise in connection with the sinking funds of local authorities or of commercial or financial undertakings, and will not be further considered.

Practical Discount as Compared with Present Value, Discount of Bills, &c. The above extracts show that £100 at 5 per cent. at the end of one year will amount to £105, and that £105 due at the end of one year at 5 per cent. is worth now £100. The difference between the two amounts viz., £5, is the mathematical or true discount, and is based upon the present value. In practical finance the method adopted in discounting bills is to deduct interest at the rate per cent. from the amount of the bill payable at the end of the period, and as this amount is always greater than the present value, practical discount, as it is called, is always greater than the mathematical or true discount. For instance, a bill for £105 due at the end of one year, and discounted by the bank at 5 per cent., is worth now £99·75, ascertained as follows:—

Amount of the bill	£105·00
Less the practical discount at 5 per cent. for one year	£5·25
or a net value of	£99·75
If the customer leaves this sum on deposit with the bank, at 5 per cent. he will at the end of the year be credited with 5 per cent. upon £99·75 or	£4·9875
and will then receive ...	£104·7375
as compared with the amount of the bill	£105·
a difference of	£0·2625

In other words, he would lose and the bank would gain £0·2625 although the bank have had the use of the money for the whole of the year.

The bank would gain the difference between the practical discount of	£5·25
and the true or mathematical discount of	£5·00
	£0·25
And in addition, interest upon this amount for one year at 5 per cent., or	£0·0125
	£0·2625

This proves that the present values as given in the tables of compound interest are not available for discounts which are merely arithmetical calculations, and for which special tables are constructed.

CHAPTER IV.

COMPOUND INTEREST AS APPLIED TO A SUM OF MONEY.

TABLE I. The amount of £1 in any number of years.

THE FORMULA, $A = P\ R^N$, AND RULES DEDUCED THEREFROM. CALCULATION BY THE ARITHMETICAL METHOD. COMPILATION OF TABLES. THOMAN'S METHOD AND FORMULA.

AUTHOR'S STANDARD CALCULATION FORM, NO. 1.

Formulæ.

A. To find the Amount of £1 in any number of years, as given in the published tables:—

Formula, $A = R^N$

by logs.: *Log. (Amount of £1)* $=$ *Log.* R^N

B. To find the Amount of any sum of money in any number of years:—

Formula, $A = P\ R^N$

by logs: *Log. (Amount of principal sum)* $=$ *Log. (principal sum)* $+$ *Log.* R^N

The above formulæ, and methods by logs. apply equally to Thoman's Formulæ and Tables, which are fully described in Chapter IX.

General Rules deduced from the above formulæ.

To find the amount of any sum of money in any number of years. *Author's Standard Calculation Form, No. 1.*

Rule 1. If the rate per cent. be not given in Table I, or in Thoman's Tables:—

Proceed by the formula relating to Table I.

Calculation (IV) 3 A.

Rule 2. If the rate per cent. be given in Table I:—

Multiply the amount given in the table, by the given sum. The product is the amount required.

Calculation (IV) 3 B.

Rule 3. *If the rate per cent. be given in Thoman's Tables:—*
To the log. of the given sum, add the log. of R^N *as given by Thoman. The sum of the logs. is the log. of the amount required.* *Calculation (IV) 3 C.*

To find the rate per cent., or number of years, proceed as shown in the standard form for the purpose, given in Chapter X.

The formula, $A = P\,R^N$, will now be applied to the solution of problems involving compound interest in relation to a sum of money, whether now in hand or payable or receivable at any future date. The published tables are as follows:—

Table I. The amount of £1 in any number of years.

Table II. The present value of £1 due at the end of any number of years.

Each table will be considered in detail to show the method of compilation by means of the above formula, but in the present case the arithmetical method of calculation will first be given in full, in order to point out the relation between the two methods.

The Arithmetical Method. In the following calculation, IV (1), at the end of the first year, interest at 5 per cent. is added to the principal sum in hand at the beginning of the year. At the end of the following, and each subsequent year, interest is added to the amount of principal and interest combined, at the beginning of the year. The amount of added interest increases each year, but if each item of interest be compared with the sum upon which it is based, it will be seen that in all cases they bear the same ratio, namely, 0·05 to 1. On comparing the amount of principal and interest at the end of any year, with the similar amount at the end of the succeeding year, it will be observed that they are always in the ratio of 1 to 1·05.

In other words, although an amount of interest has been *added* each year, the amount of principal and interest at the end of each year might have been obtained by *multiplying* the amount at the end of the previous year by 1·05, or the ratio R. This is therefore a geometrical progression increasing at a ratio of 1·05. This calculation will be referred to again in Chapter VI, when considering the derivation of the formula relating to an annual or other periodic payment, and the discussion of the matter in that chapter may be referred to at this stage with advantage.

Calculation (IV) 1.

To find the Amount of a given Sum at the end of a given term.
Table I.

Required the Amount of £1 at the end of 5 years at 5 per cent., compound interest.

By Arithmetical Calculation.

Principal Sum at the beginning of the first year... ...		1·0000
1. First year's Interest thereon	$=(r)$	·0500
$(1 \times 1{\cdot}05)$	$=(R)$	1·0500
2. Second year's Interest thereon		·0525
$(1{\cdot}05 \times 1{\cdot}05)$		1·1025
3. Third year's Interest thereon		·0551
$(1{\cdot}1025 \times 1{\cdot}05)$		1·1576
4. Fourth year's Interest thereon		·0579
$(1{\cdot}1576 \times 1{\cdot}05)$		1·2155
5. Fifth year's Interest thereon		·0608
$(1{\cdot}2155 \times 1{\cdot}05)$		1·2763

which is the required amount at the end of the 5th year; and agrees with the amount given in Table I. A further amplification of this calculation will be made in Chapter VI.

The Mathematical Method. It very rarely happens that calculations of compound interest are required for so short a period as 5 years; generally they are for very much longer periods. Consequently the arithmetical method as shown in the above Calculation (IV) 1, becomes cumbrous and liable to error, and it is imperative to adopt a shorter method, namely, the algebraical or mathematical one, based upon the formula, $A = P\ R^N$. Here it is required to find the amount A, knowing that:—

$$P = 1,\ R = 1{\cdot}05 \text{ and } N = 5.$$

The equation therefore becomes:

$$A = R^N, \text{ or } A = (1{\cdot}05)^5.$$

but to raise R, or 1·05 to the 5th power or perhaps to the 20th, 30th, or 60th power is a much longer task than to make the

original calculation by the arithmetical method, as in the previous example, and recourse is had to logarithms, which have been fully described in Chapter II. The calculation will be made upon standard calculation form No. 1 by method (A) therein contained, and it will be found that the resulting amount agrees with the value given in Table I in the published tables. It will also be seen that the resulting log. of the required amount agrees with the log. of R^N in Thoman's tables.

The above methods will now be applied to the following example in order to demonstrate that the calculation by means of logarithms and the above formula is quite as simple, not only for any longer period, but at any rate per cent., whereas the calculation by the arithmetical method will be longer in proportion to the number of years, and will consequently involve a greater possibility of error in the arithmetical computation.

"Required the amount of £500 at the end of 20 years at 5 per cent. per annum compound interest." Calculation (IV) 3.

As in the previous calculation, relating to £1 only, the result will be ascertained by the same methods, viz:—

A. by the formula, $A = P\,R^N$ ... ... Rule 1

B. by the published table No. I, giving the amount of £1 at the end of any number of years ... ... Rule 2

C. by Thoman's tables ... ... Rule 3

in each case adopting the logarithmic method of calculation. The above rules and formulæ are fully set out in the heading to this chapter.

Thoman's Method and Formula. Although Thoman's method applies more particularly to calculations involving annuities or other periodic payments, these tables may with advantage be utilised to solve problems relating to the amount and present value of £1, owing to the fact that the actual logs. of R^N are there given, instead of having to be taken from the log. tables. The full consideration of Thoman's tables is contained in Chapter IX.

Calculation (IV) 2.

Standard Calculation Form, No. 1.

To find the future amount of a present sum, and thereby prove the accuracy of the published table. Table I.

Required the amount of £1 at the end of 5 years at 5 per cent., per annum, compound interest.

(A) By Formula. $A=P\,R^N$ Rule 1, Chapter IV.

Log. R^N { Log. Ratio	R	1·05	0·0211893
multiply Log. R by	N	5	5
	R^N	$(1·05)^5$	0·1059465
Log. Present Sum	P	1·	0·
add Log. R^N above	R^N		0·1059465
	A		0·1059465

Required future amount, £1·27628, which agrees with the result obtained by the arithmetical method, Calculation (IV) 1, and also with the amount given in Table I.

(B) By Table I. $A=P\,R^N$ Rule 2, Chapter IV.

Table I. 5 years, 5 per cent.		
Amount of £1	R^N	1·27628
add Log. Present Sum	P	
	A	1·27628

Required future amount, £1·27628, as given in Table I.

(C) By Thoman's Table. $A=P\,R^N$ Rule 3, Chapter IV.
5 per cent. 5 years.

Log. Present Sum	P	1·	0·
add Log. R^N	R^N		0·1059465
	A		0·1059465

Required future amount, £1·27628. This log. is given in Thoman's Table.

Calculation (IV) 3.

Standard Calculation Form, No. 1.

To find the future amount of a present sum. Table I.

Required the amount of £500 at the end of 20 years at 5 per cent. per annum, compound interest.

(A) By Formula.	$A = P R^N$	Rule 1, Chapter IV.	
Log. R^N: Log. Ratio	R	1·05	0·0211893
multiply Log. R by	N	20	20
	R^N	$(1·05)^{20}$	0·4237860
Log. Present Sum	P	500	2·6989700
add Log. R^N above	R^N		0·4237860
	A		3·1227560

Required future amount, £1326·65.

(B) By Table I.	$A = P R^N$	Rule 2, Chapter IV.	
Table I. 20 years, 5 per cent.			
Amount of £1	R^N	2·6533	0·4237860
add Log. Present Sum	P	500	2·6989700
	A		3·1227560

Required future amount, £1326·65.

(C) By Thoman's Table. 5 per cent. 20 years.	$A = P R^N$	Rule 3, Chapter IV.	
Log. Present Sum	P	500	2·6989700
add Log. R^N	R^N		0·4237860
	A		3·1227560

Required future amount, £1326·65.

CHAPTER V.

COMPOUND INTEREST AS APPLIED TO A SUM OF MONEY (*Continued*).

TABLE II. The present value of £1, due at the end of any number of years.

DERIVATION OF THE FORMULA, $P=\frac{A}{R^N}$, AND RULES DEDUCED THEREFROM. COMPILATION OF TABLES. TABLE OF RATIOS, AND LOGS. OF R, AND r. CALCULATIONS FOR PERIODS OTHER THAN YEARS. THOMAN'S METHOD AND FORMULA.

AUTHOR'S STANDARD CALCULATION FORM, No. 2.

Formulæ.

A. To find the present value of £1, due at the end of any number of years, as given in the published tables:—

Formula, $$P=\frac{1}{R^N}$$

by logs.: *Log.* (*present value of £1*) = *Log.* 1(=0) − *Log.* R^N

B. To find the present value of any sum of money, due at the end of any number of years:—

Formula, $$P=\frac{A}{R^N}$$

by logs.: *Log.* (*present value*) = *Log.* (*amount due at end of period*) − *Log.* R^N

The above formulæ, and methods by logs., apply equally to Thoman's formulæ and tables, which are fully described in Chapter IX.

General Rules deduced from the above formulæ.

To find the present value of any sum of money, due at the end of any number of years.

Author's Standard Calculation Form, No. 2.

This formula, $P = \frac{1}{R^N}$ may now be used to ascertain the amounts given in Table II, and, as in the previous example, the calculation will be made by three different methods, namely,

A, by formula ... ... Rule 1.

B, by the published Table II, giving the present value of £1 due at the end of any number of years ... ... Rule 2.

C, by Thoman's Tables ... ... Rule 3.

in each case adopting the logarithmic method of calculation. The above rules and formulæ are fully set out in the heading of this chapter.

THOMAN'S METHOD AND FORMULA. In considering the methods of finding the amounts of £1 in any number of years as given in Table I, attention was drawn to the advantage of using Thoman's tables. It was found that the calculation by this method is similar to the calculation by Table I, but in the case of Table II, relating to the present value of a future sum, it is necessary to make use of the reciprocal of R^N, or $\frac{1}{R^N}$. The only difference between the two tables is that in the case of Table I the log. of R^N is added to the log. of the present sum, whereas in the case of Table II the same log. is deducted from the log. of the future given sum of which it is required to find the present value. Calculation (V) 1.

The same formula will next be used in order to ascertain the present value of £1326·65 due at the end of 20 years at 5 per cent. compound interest, and thereby prove the converse of Calculation (IV) 3, adopting the same methods, namely, by formula: by the published Table II: and by Thoman's method. Calculation (V) 2.

CALCULATIONS FOR PERIODS OTHER THAN YEARS. In cases where it is required to calculate compound interest for periods other than years, and the rate per cent. is expressed as per annum, it is necessary to take a rate per cent. proportionate to the period of a year. For instance, if it be required to calculate the amount of a sum of money rolling up half yearly, *double the number of years and take one-half the rate per cent. per annum*, as follows:—

£1 at the end of 10 years at 10 per cent. per annum will amount to (yearly breaks) ... ... £2·5937

£1 at the end of 10 years at 10 per cent. per annum will amount to (half-yearly breaks) =20 years at 5 per cent. ... ... £2·6533

£1 at the end of 10 years at 10 per cent. per annum will amount to (quarterly breaks) =40 years at $2\frac{1}{2}$ per cent. ... ... £2·6851

This is a very useful method to adopt when it is required to ascertain the effect of compounding the interest at various periods, and the rule applies equally to calculations involving annuities. All that is necessary is to deal with the number of periods at the corresponding rate per cent. per period, based upon the rate per cent. per annum.

The Factors R (Ratio) and r (The Interest of £1 for One Year) and the Corresponding Logarithms.

In order to simplify the method by formulæ and logs. the following Table, No. V. A., has been prepared. It gives the ratio (R) and the corresponding logs. for 49 rates from $\frac{1}{4}$ per cent. to 7 per cent. It also contains the values and corresponding logs. of the factor (r) which is the interest upon £1 for one year. There is not anything difficult in the compilation of the table, which is here given only for convenience of reference. The logarithms corresponding to any rate per cent. may be ascertained from the log. tables at the time of making the calculation, but since many of the ratios contain six figures, it involves the use of the proportional parts of the logarithms, and a reference to this table will save time. When dealing with annuities, the logs. of (R^N) and (r) are required in each calculation, and as they have always to be looked for in different parts of the log. tables, it is a convenience to have them in one place. If it is necessary to make a calculation at any intermediate rate per cent. not included in this table all that is required is to find the ratio, which is *one pound, increased by interest upon one pound, for one year, at the given rate per cent.,* and then the corresponding log. The factor (r), as will be seen from the table, is ascertained by deducting 1 from the ratio so found. The logs. of both are found from the tables of logs. in the usual way, paying due attention to the sign of the "characteristic" of the log. of (r). The logs. of (R^N) are given in Thoman's tables for many rates per cent. for a large number of years.

Calculation (V) 1.

Standard Calculation Form, No. 2.

To find the present value of a sum due at the end of any number of years, and thereby prove the accuracy of the published table. Table II.

Required the present value of £1, due at the end of 20 years, at 5 per cent. per annum compound interest.

(A) By Formula. $P=\frac{A}{R^N}$ Rule 1, Chapter V.

Log. R^N { Log. Ratio	R	1·05	0·0211893
multiply Log. R by	N	20	20
	R^N	$(1·05)^{20}$	0 4237860
Log. Future Sum	A	1·	0·
deduct Log. R^N above	R^N		0·4237860
	P		$\bar{1}$·5762140

Required present value, £0·37689, which agrees with the amount given in Table II.

(B) By Table II. $P=\frac{A}{R^N}$ Rule 2, Chapter V.

Table II. 20 years, 5 per cent. Present value of £1	$\frac{1}{R^N}$	0·37689
add Log. Future Sum	A	
	P.	0·37689

Required present value, £0·37689, as given in Table II.

(C) By Thoman's Table. $P=\frac{A}{R^N}$ Rule 3, Chapter V.

5 per cent. 20 years.

Log. Future Sum	A	1·	0·
deduct Log.	R^N		0·4237860
	P		$\bar{1}$·5762140

Required present value, £0·37689. This log. is given in Thoman's Table.

Calculation (V) 2.

Standard Calculation Form, No. 2.

To find the present value of a sum due at the end of any number of years. Table II.

Required the present value of £1326·65, due at the end of 20 years, at 5 per cent. per annum compound interest.

(A) By Formula. $P = \frac{A}{R^N}$ Rule 1, Chapter V.

Log. R^N { Log. Ratio	R	1·05	0·0211893
multiply Log. R by	N	20	20
	R^N	$(1·05)^{20}$	0·4237860
Log. Future Sum	A	1326·65	3·1227560
deduct Log. R^N above	R^N		0·4237860
	P		2·6989700

Required present value, £500·00.

(B) By Table II. $P = \frac{A}{R^N}$ Rule 2, Chapter V.

Table II. 20 years, 5 per cent. Present value of £1	$\frac{1}{R^N}$	0·37689	$\bar{1}$·5762140
add Log. Future Sum	A	1326·65	3·1227560
	P		2·6989700

Required present value, £500·00.

(C) By Thoman's Table. $P = \frac{A}{R^N}$ Rule 3, Chapter V.

5 per cent. 20 years.

Log. Future Sum	A	1326·65	3·1227560
deduct Log.	R^N		0·4237860
	P		2·6989700

Required present value, £500·00.

TABLE V, A.

Giving the values of (R) and (r) for the following rates per cent. (from ¼ to 7 per cent.) and the corresponding log of each value.

(R) = the amount of £1 plus one year's interest at any rate per cent.

$$= (1+r)$$

(r) = the interest upon £1 for one year at any rate per cent.

$$= (R-1).$$

The Logarithms of (R^N) are given in Thoman's Tables under each rate per cent.

Rate	Ratio = R		Rate	Interest on £1 for 1 year = r	
%	R	Log. R	%	r	Log. r
¼	1·0025	0·00108438	¼	0·0025	$\bar{3}$·3979400
½	1·005	0·00216606	½	0·005	$\bar{3}$·6989700
¾	1·0075	0·00324505	¾	0·0075	$\bar{3}$·8750613
1	1·01	0·00432137	1	0·01	$\bar{2}$·0000000
½	1·015	0·00646604	½	0·015	$\bar{2}$·1760913
⅝	1·01625	0·00700056	⅝	0·01625	$\bar{2}$·2108534
¾	1·0175	0·00753442	¾	0·0175	$\bar{2}$·2430380
⅞	1·01875	0·00806762	⅞	0·01875	$\bar{2}$·2730013
2	1·02	0·00860017	2	0·02	$\bar{2}$·3010300
⅛	1·02125	0·00913207	⅛	0·02125	$\bar{2}$·3273589
¼	1·0225	0·00966332	¼	0·0225	$\bar{2}$·3521825
⅜	1·02375	0·01019391	⅜	0·02375	$\bar{2}$·3756636
½	1·025	0·01072387	½	0·025	$\bar{2}$·3979400
⅝	1·02625	0·01125317	⅝	0·02625	$\bar{2}$·4191293
¾	1·0275	0·01178183	¾	0·0275	$\bar{2}$·4393327
⅞	1·02875	0·01230985	⅞	0·02875	$\bar{2}$·4586378
3	1·03	0·01283722	3	0·03	$\bar{2}$·4771213
⅛	1·03125	0·01336396	⅛	0·03125	$\bar{2}$·4948500
¼	1·0325	0·01389006	¼	0·0325	$\bar{2}$·5118834
⅜	1·03375	0·01441552	⅜	0·03375	$\bar{2}$·5282738
½	1·035	0·01494035	½	0·035	$\bar{2}$·5440680
⅝	1·03625	0·01546454	⅝	0·03625	$\bar{2}$·5593080

Rate	Ratio = R		Rate	Interest on £1 for 1 year = r	
%	R	Log. R	%	r	Log. r
¾	1·0375	0·01598811	¾	0·0375	$\bar{2}$·5740313
⅞	1·03875	0·01651104	⅞	0·03875	$\bar{2}$·5882717
4	1·04	0·01703334	4	0·04	$\bar{2}$·6020600
⅛	1·04125	0·01755501	⅛	0·04125	$\bar{2}$·6154240
¼	1·0425	0·01807606	¼	0·0425	$\bar{2}$·6283889
⅜	1·04375	0·01859649	⅜	0·04375	$\bar{2}$·6409781
½	1·045	0·01911629	½	0·045	$\bar{2}$·6532125
⅝	1·04625	0·01963547	⅝	0·04625	$\bar{2}$·6651117
¾	1·0475	0·02015403	¾	0·0475	$\bar{2}$·6766936
⅞	1·04875	0·02067197	⅞	0·04875	$\bar{2}$·6879746
5	1·05	0·02118930	5	0·05	$\bar{2}$·6989700
⅛	1·05125	0·02170601	⅛	0·05125	$\bar{2}$·7096939
¼	1·0525	0·02222210	¼	0·0525	$\bar{2}$·7201593
⅜	1·05375	0·02273759	⅜	0·05375	$\bar{2}$·7303785
½	1·055	0·02325246	½	0·055	$\bar{2}$·7403627
⅝	1·05625	0·02376672	⅝	0·05625	$\bar{2}$·7501225
¾	1·0575	0·02428038	¾	0·0575	$\bar{2}$·7596678
⅞	1·05875	0·02479342	⅞	0·05875	$\bar{2}$·7690079
6	1·06	0·02530587	6	0·06	$\bar{2}$·7781513
⅛	1·06125	0·02581770	⅛	0·06125	$\bar{2}$·7871061
¼	1·0625	0·02632894	¼	0·0625	$\bar{2}$·7958800
⅜	1·06375	0·02683957	⅜	0·06375	$\bar{2}$·8044802
½	1·065	0·02734961	½	0·065	$\bar{2}$·8129134
⅝	1·06625	0·02785904	⅝	0·06625	$\bar{2}$·8211859
¾	1·06750	0·02836788	¾	0·06750	$\bar{2}$·8293038
⅞	1·06875	0·02887613	⅞	0·06875	$\bar{2}$·8372727
7	1·07	0·02938378	7	0·07	$\bar{2}$·8450980

CHAPTER VI.

COMPOUND INTEREST AS APPLIED TO AN ANNUAL OR OTHER PERIODIC PAYMENT.

TABLE III. The amount of £1 per annum in any number of years.

GENERAL REMARKS AS TO ANNUITIES. THE RELATION BETWEEN THE AMOUNTS OF £1, AND OF £1 PER ANNUM. THE ARITHMETICAL METHOD FURTHER CONSIDERED. DERIVATION OF THE FORMULA, $M = Ay\left(\frac{R^N - 1}{r}\right)$ AND RULES DEDUCED THEREFROM. COMPILATION OF TABLES. THOMAN'S METHOD AND FORMULA.

AUTHOR'S STANDARD CALCULATION FORM, No. 3.

Formulæ.

A. To find the amount of £1 per annum in any number of years, as given in the published tables:—

(1) *Formula,* $$M = \left(\frac{R^N - 1}{r}\right)$$

by logs.: *Log.* (*amount of £1 per annum*) = *Log.* $(R^N - 1)$ − *Log.* r

(2) *By Thoman's method:—*

Formula, $$M = \frac{R^N}{a^n}$$

by logs.: *Log.* (*amount of £1 per annum*) = *Log.* $R^N + 10$ − *Log.* a^n

B. To find the amount of any annuity in any number of years:—

(1) *Formula,* $$M = Ay\left(\frac{R^N - 1}{r}\right)$$

by logs.: *Log.* (*amount of annuity*) = *Log.* *annuity* + *Log.* $(R^N - 1)$ − *Log.* r

(2) *By Thoman's method:—*

Formula, $$M = Ay\frac{R^N}{a^n}$$

by logs.: *Log.* (*amount of annuity*) = *Log.* *annuity* + *Log.* $R^N + 10$ − *Log.* a^n

The present chapter deals only with the formula $M = Ay\left(\frac{R^N - 1}{r}\right)$. *Thoman's method and formulæ, are fully described in Chapter IX.*

General Rules deduced from the above formulæ.

To find the amount of any annuity in any number of years.
Author's Standard Calculation Form, No. 3.

Rule 1. If the rate per cent. be not given in Table III, or in Thoman's Tables:—
Proceed by the formula relating to Table III.
Calculation (VI) 2 A.

Rule 2. If the rate per cent. be given in Table III:—
Multiply the amount given in the table, by the given annuity. The product is the amount required.
Calculation (VI) 2 B.

Rule 3. If the rate per cent. be given in Thoman's Tables:—
To the log. of the given annuity, add the log. of R^N *as given by Thoman. Add 10 to the sum of the two logs., and deduct therefrom the log. of* a^n *as given by Thoman. The remainder is the log. of the required amount.* *Calculation (VI) 2 C.*

To find the rate per cent., or number of years, proceed as shown in the standard form for the purpose, given in Chapter X.

Annuities or other Periodic Payments. All problems relating to annual sums involve calculations of a more complex character than the steady accumulation of a given sum of money. Matters are complicated by the intrusion of a factor representing an equal annual or other periodic sum, to be set aside, received or paid, at the end of each year, and accumulated at a given rate per cent., for a given number of periods. Such an equal annual or other periodic sum is called an annuity, but in this connection it should be borne in mind that actuarially the term annuity includes any definite sum of money to be paid or received at the end of any given number of regular intervals. There is room for a better word, but it does not matter so long as it is known what the term includes. In the following pages the word annuity will be used to denote any equal sum payable at the end of regular periods, except that in the case of sinking funds, the word "instalment" or "annual increment" will be substituted.

As in the case of a principal sum, an annuity or other periodic payment may be expressed in terms of its "amount" or "present value" which are given in Tables III and IV respectively.

THE FACTORS R (Ratio) AND r (the Interest of £1 for One Year).

In all calculations involving a geometrical progression the predominant factor is the ratio which, in the algebraical formula, is expressed by the symbol, r. A pure geometrical progression relates only to a series of numbers, increasing in a definite ratio, similar to the annual accumulation, by way of compound interest, of a given sum of money as described in Chapter IV, dealing with Table I. The algebraical formula for a pure geometrical progression does not provide for any further addition to each term of the progression. In the case of compound interest, however, the problem may be complicated by the annual or other periodic addition of a definite sum, namely, the annuity, and it is necessary therefore to amend the formula, $A = P\ R^N$, by dividing the factor, R, or ratio, into two parts, namely, the actual algebraical ratio and the equal annual addition to each term of the progression, representing the constant sum or annuity to be added to each term. In the algebraical formula the ratio is expressed by the symbol, r In the formula relating to compound interest two symbols are used, namely:—

R = the common ratio existing between the successive terms of the progression irrespective of any periodic equal additions to the progression. This factor, R, in the formulæ relating to compound interest is the equivalent of the algebraical factor (r).

r = the annual or other periodic sum added to each term of the progression, and which, as regards the formulæ relating to unity, represents the annual interest of £1 for one year.

In this manner the accumulation of an annual sum by way of compound interest, cannot properly be considered a pure geometrical progression. It is rather the sum of several arithmetical progressions in echelon, which accounts for the difficulty in determining the rate per cent. by means of the formula, as will be seen on reference to the standard form for the purpose given in Chapter X.

The Relation between the Amount of £1 and of £1 per Annum. It is necessary to derive a formula relating to the amounts and the present values of £1 per annum as given in the published tables, which formula, although based thereon, is of a somewhat more complicated character than the simple formula relating to Tables I and II. The additional symbols which will be required have already, in anticipation, been explained in Chapter III.

Before proceeding to find such a formula the subject will be considered from the point of view of the accumulation of a single sum now in hand, as illustrated by Calculation (IV) 1 in Chapter IV. It is possible to ascertain the sum to which an annuity will amount at the end of a stated period, by treating each of the annual payments separately, and finding the sums to which they will respectively amount at the end of the period, by the method already considered in relation to Table I. The total of these separate results will represent the sum to which the whole annuity will amount at the end of the period (see columns 1 to 4 in the following table).

The method is a cumbrous one, and therefore not practical, but the working of such a calculation is given in order to demonstrate the relation between Table I, giving the amounts of £1, and Table III, giving the amounts of £1 per annum. It will also emphasise what has been already pointed out in Chapter III, namely, the difference between the amounts of £1 and of £1 per annum at the end of any equal number of years, as also shown in columns 5 and 6 in the following table. This difference is due to the fact that in all calculations of this nature the sum of money of which it is required to find the amount at the end of a term of years, as in Table I, is assumed to be in hand and to commence to accumulate at once, whereas, in the case of an annuity, the annual or other periodic payments are assumed to be made at the end of each year or period, at which date they begin to accumulate. An annuity of £1 for a given number of years may, therefore, be considered as a series of sums of money, each of which is deferred, both as to the date of payment and of accumulation, for 1, 2, 3, 4, etc., years. Taking as an example an annuity of £1 for 10 years to accumulate at 5 per cent. per annum, the sum to which each separate payment will amount at the end of the 10 years will be ascertained by the method adopted in Calculation (IV) 1, and after deriving the formula relating to the amounts of an annuity, as given in Table III, the same example will be worked out, by means of the formula, in Calculation (VI) 1.

TABLE VI, A.

Showing the method of finding the amount of an annuity from the figures given in Table I, relating to a principal sum and illustrating the relation between the amounts of £1 and of £1 per annum at the end of one year.

Rate of accumulation, 5 per cent.

		TABLE I.		TABLE III.		TABLE I.		
Amount set aside at end of year.	Will accumulate for years	Amount of each annual sum at end of 10th year.	Total at end of each year of the amounts in Col. 3.		Amount of £1 per annum at end of each year from 1 to 10 years.		Amount of £1 at end of each year from 1 to 10 years.	Total at end of each year of the amounts in Col. 6.
(1)	(2)	(3)	(4)		(5)		(6)	(7)
10	0	1·0000	1·0000	1	1·0000		—	—
9	1	1·0500	2·0500	2	2·0500	1	1·0500	1·0500
8	2	1·1025	3·1525	3	3·1525	2	1·1025	2·1525
7	3	1·1576	4·3101	4	4·3101	3	1·1576	3·3101
6	4	1·2155	5·5256	5	5·5256	4	1·2155	4·5256
5	5	1·2763	6·8019	6	6·8019	5	1·2763	5·8019
4	6	1·3401	8·1420	7	8·1420	6	1·3401	7·1420
3	7	1·4071	9·5491	8	9·5491	7	1·4071	8·5491
2	8	1·4775	11·0266	9	11·0266	8	1·4775	10·0266
1	9	1·5513	12·5779	10	12·5779	9	1·5513	11·5779
		12·5779					11·5779	

In the above table:—

Column 1, contains the year at the end of which each annual sum is set aside, and *Column 2* the number of years for which it afterwards accumulates.

Column 3, is taken item by item from Table I (with the exception of the first item of £1) and shows the amount of each separate annual sum (beginning with the 10th) at the end of the 10th year as if it were accumulated separately. The total of Column 3 is the accumulated amount of the whole of the annual sums obtained in this manner, and agrees with the amount given in Table III and found by Calculation (VI) 1.

Column 4, gives, at the end of each successive year, the total of the previous items in Column 3, which is the amount of all the annual sums set aside up to the end of that

year. The items in this column correspond, year by year, with the amounts of an annuity of £1 given in *Column 5*, which is copied item for item, from Table III, which gives the amounts of an annuity of £1 for any number of years.

Column 6, contains the amounts of £1 at the end of each year from 1 to 10 years, copied, item by item, from Table I. These figures correspond with each item, except the first, in Column 3.

Column 7, contains the total at the end of each successive year of the previous items in Column 6, and might have been obtained by adding together the figures given in Table I. On comparing the totals of Columns 3 and 6, it will be seen that the total of Column 6, at the end of the 10th year, is less by £1 than the total of Column 3. Similarly, if at the end of any year the totals in Column 5 are compared with the totals in Column 7, the same difference will be found.

Consequently, if it be required to ascertain the accumulated amount of an annuity of £1 for 10, or any other number of years, at 5 per cent. per annum from Table I, which gives the amounts of £1, it may be found by adding together the successive amounts given in Table I for 9, or one less than the specified number of years, and increasing the sum so obtained by £1. The sum of the 9 amounts is the amount of nine years' accumulation of £1 per annum, and the £1 so added is the last annual sum, which does not accumulate at all owing to its being set aside on the last day of the last year of the term.

Derivation of the Formula. The above-described arithmetical method of finding the amount of an annuity for any number of years depends upon treating each annual sum as a separate entity, but does not treat the annuity quâ annuity, and, further, it does not give any clue to a rule or formula by which the result may be obtained by direct mathematical calculation. Many problems contain factors involving the accumulation of £1, and also of £1 per annum, and it is advisable therefore that all formulæ should be expressed in the same or similar terms. The formula relating to the amount of £1 in any number of years, namely, $A = P R^N$, has already been ascertained, and it will now be used in order to deduce therefrom a formula relating to the accumulation of periodic sums. The practical application of that formula will be first

considered, and will be based upon the arithmetical Calculation (IV) 1, as afterwards proved by means of the formula, in Calculation (IV) 2. As explained in Chapter IV, at the end of the first year, interest at 5 per cent. per annum was added to the original principal sum of £1, and at the end of each subsequent year interest at 5 per cent. per annum was added to the amount of principal and interest at the beginning of such year. In Calculation (IV) 1, the interest added each year was treated as one sum, and was not divided in order to differentiate between the interest added yearly in respect of the original principal sum as distinguished from the interest added yearly upon the interest added in previous years. In the following table (No. VI, B) such a distinction has been made, and the results obtained in Calculation (IV) 1 are repeated in Column 2. The interest added each year has been divided as between the principal and the interest previously added, and Column 3 contains the constant annual amount of interest upon the original principal of £1, which is (r) in the list of symbols given in Chapter III. Columns 4, 5, 6, and 7 contain each year's accumulated interest upon each annual amount of interest (r) upon the original principal of £1. The table is as follows:—

TABLE VI, B.

Showing the amount of £1, for 5 years at 5 per cent. per annum. Calculation (IV) 1. Showing also the annuity of $(r) = 0{\cdot}05$, and its accumulations.

1 At end of year.	2 Amount of £1. 5 years 5 %.	3 Annual Interest on £1 Principal.	4 Accumulation of (r) Annual Interest on £1 at end of 2nd	5 3rd	6 4th	7 5th	8 Total Accumulation of (r).
	1·0000						
1	·0500	·0500	·0025	·0026	·0028	·0029	·0108
	1·0500						
2	·0525	·0500	—	·0025	·0026	·0028	·0079
	1·1025						
3	·0551	·0500	—	—	·0025	·0026	·0051
	1·1576						
4	·0579	·0500	—	—	—	·0025	·0025
	1·2155						
5	·0608	·0500	—	—	—	—	—
	1·2763	·2500	·0025	·0051	·0079	·0108	·0263

1·0000 Original Sum
·2500 Annual Interest
·0263 Accumulations of Annual Interest

The original principal sum of £1 may now be left out of the calculation, and be considered only as the origin of an annual sum or annuity, of £0·05 to be accumulated for 5 years at 5 per cent. per annum compound interest. It is in fact, at 5 per cent, the present value of a perpetual annuity of £0·05. The results obtained in the above table will now be translated into terms of the formula $A = P\ R^N$, writing against each factor in the arithmetical result the corresponding symbol in the formula, but as the formula is being considered in its relation to £1 only, there will be substituted for P its equivalent 1, with the following result, viz., $A = R^N$.

The above Table VI, B, expressing the results of Calculation IV (1) may be analysed as follows:—

	Actual results.	Formula.
Amount of £1 in 5 years at 5 per cent. ...	1·2763	R^N
Deduct, the principal sum of which this is the amount at the end of 5 years (Table I)	1·	1
leaving ...	0·2763	$R^N - 1$
which is the accumulated amount of the annual interest upon £1 at 5 per cent., or 1·05 − 1 = 0·05	0·05	$R - 1$

which is the annuity which will in 5 years at 5 per cent., amount to £0·2763, as shown in the above table, No. VI, B.

The formula relating to the accumulation of an annual sum is derived from the foregoing results as follows:—

It has been ascertained by means of the formula $A = P\ R^N$ relating to the accumulation of £1 as given in Table I, that the amount (A), of £1 (P) at the end of any number of years is	R^N
and by deducting therefrom the original sum, P, or its equivalent, which in this case is	1
a constant is obtained which will apply to any rate per cent., namely	$R^N - 1$

This constant represents the accumulated amount of the annual interest upon £1, resulting from the accumulation of the original principal sum P, at the ratio R, for N years.

In the above example, the ratio, which is P, plus one year's interest, is	R or 1·05
and by deducting therefrom the original principal sum P, or 1, the remainder is	R − 1 or 0·05

which represents the interest upon £1 for one year and is constant for any rate per cent.

Expressing the above in terms of the calculation in Table VI, B, it is found that:—

$(R^N - 1)$ or 0·2763 is the accumulated amount of an annual sum of $(R - 1)$ or 0·05 for (N) or 5 years, at a ratio (R) 1·05, which is the equivalent of 5 per cent. per annum.

Stated in the form of a proportion the problem becomes:—

If 0·05 per annum or $(R - 1)$ amounts to 0·2763 or $(R^N - 1)$, what sum will £1 per annum amount to under the same conditions, as follows:—

$$\frac{0{\cdot}2763}{0{\cdot}05} = \frac{R^N - 1}{R - 1} = 5{\cdot}526$$

which agrees with the amount given in Table III, and provides a formula which may be used to calculate the amount of £1 per annum for any number of years at any rate per cent. To find the amount of any other annual sum all that is required is to multiply the result obtained in the above manner by the annual sum in question.

It is not possible to simplify the above factor $(R^N - 1)$ because R^N varies with each number of years, but $(R - 1)$ may be expressed by a simple symbol because it is always constant for each rate per cent. It may be found by deducting unity from R or by dividing the rate per cent. by 100. The factor $(R - 1)$ is denoted by the symbol (r) to show at once its relation to, and variation from, the factor (R) from which it is derived.

The amount of £1 per annum is denoted by the symbol (M) to distinguish it from (A) the amount of £1.

The formula therefore becomes:—

(1) as to £1 per annum:

$$M=\left(\frac{R^N-1}{r}\right)$$

and

(2) as to any annual sum (Ay):

$$M=Ay\left(\frac{R^N-1}{r}\right)$$

and the symbols have the meanings described in Chapter III.

The annuity or other periodic sum in all cases, as already pointed out, is presumed to be paid or received, set aside or invested, at the end of the first and each succeeding year, which is the usual method in all annuity calculations. If it be set aside at the beginning of the year the calculation is somewhat different.

CALCULATIONS. Having found the above formula relating to the amount of £1 per annum in any number of years, two calculations will now be made by its aid, upon the author's standard form No. 3. Both will include the three methods of which the general rules are stated at the head of this chapter, namely, by formula, by the published tables, and by Thoman's method and tables. The first calculation will deal only with an annuity of £1, and will show the method of computing the amounts given in Table III, Calculation (VI) 1. The second calculation will deal with an annuity of stated amount, and will illustrate the method to be adopted in actual practice. Calculation (VI) 2.

Calculation (VI) 1.

Standard Calculation Form, No. 3.

To find the amount of an annuity in any number of years, and thereby prove the accuracy of the published table.

Table III.

Required the amount of £1 per annum for 10 years at 5 per cent. per annum compound interest.

(A) By Formula. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 1, Chapter VI.

Log. $R^N - 1$	Log. Ratio	R	1·05	0·0211893
	multiply Log. R by	N	10	10
		R^N	$(1·05)^{10}$	0·2118930
	Convert Log. to ordinary number	R^N	1·6289	
	deduct unity	−1	1·	
	Log. of this is	$R^N - 1$	0·6289	$\bar{1}$·7985779
	Log. Annuity	Ay	1·	0·0000000
	add Log. $R^N - 1$ above	$R^N - 1$		$\bar{1}$·7985779
		$Ay(R^N - 1)$		$\bar{1}$·7985779
	deduct Log. r	r	·05	$\bar{2}$·6989700
		M		1·0996079

Required future amount, £12·5779.

(B) By Table III. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 2, Chapter VI.

Table III. 10 years, 5 per cent. Amount of £1 per annum	$\frac{R^N - 1}{r}$	12·5779	
Add Log. Annuity	Ay		
	M		

Required future amount, £12·5779. This amount is given in Table III.

(C) By Thoman's Table. $M = Ay\left(\frac{R^N}{a^n}\right)$ Rule 3, Chapter VI.

5 per cent. 10 years.

Log Annuity	Ay	1·	0·0000000
Add Log. R^N in Table + 10	R^N		10·2118930
	$Ay\,R^N$		10·2118930
deduct Log. a^n	a^n		9·1122851
	M		1·0996079

Calculation (VI) 2.

Standard Calculation Form, No. 3.

To find the amount of an annuity in any number of years.

Table III.

Required the amount of £500 per annum for 10 years at 5 per cent. per annum compound interest.

(A) By Formula. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 1, Chapter VI.

Log. $R^N - 1$	Log. Ratio	R	1·05	0·0211893
	multiply Log. R by	N	10	10
		R^N	$(1·05)^{10}$	0·2118930
	Convert Log. to ordinary number	R^N	1·6289	
	deduct unity	−1	1·	
	Log. of this is	$R^N - 1$	0·6289	$\bar{1}$·7985779
	Log Annuity	Ay	500	2·6989700
	add Log. $R^N - 1$ above	$R^N - 1$		$\bar{1}$·7985779
		$Ay(R^N - 1)$		2·4975479
	deduct Log. r	r		$\bar{2}$·6989700
		M		3·7985779

Required future amount, £6288·94

(B) By Table III. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 2, Chapter VI.

Table III. 10 years, 5 per cent. Amount of £1 per annum	$\frac{R^N - 1}{r}$	12·5779	1·0996079
add Log. Annuity	Ay	500	2·6989700
	M		3·7985779

Required future amount, £6288·94

(C) By Thoman's Table. $M = Ay\left(\frac{R^N}{a^n}\right)$ Rule 3, Chapter VI.

5 per cent. 10 years.

Log Annuity	Ay	500	2·6989700
Add Log. R^N in Table + 10	R^N		10·2118930
	$Ay\ R^N$		12·9108630
deduct Log. a^n	a^n		9·1122851
	M		3·7985779

Required future amount, £6288·94

CHAPTER VII.

COMPOUND INTEREST AS APPLIED TO AN ANNUAL OR OTHER PERIODIC PAYMENT (*Continued*).

TABLE IV. The present value of £1 per annum for any number of years.

$$P = Ay\left(\frac{R^N - 1}{R^N r}\right)$$

FORMULÆ USED IN CALCULATIONS AND RULES DEDUCED THEREFROM. DERIVATION OF FORMULA AND APPLICATION TO COMPILATION OF TABLES AND TO CALCULATIONS. CALCULATIONS TO DEMONSTRATE THE THEORETICAL CONCLUSIONS BOTH AS REGARDS THE PUBLISHED TABLES AND PRACTICAL EXAMPLES. THOMAN'S METHOD AND FORMULA.

AUTHOR'S STANDARD CALCULATION FORM, NO. 4.

Formulæ.

A. To find the present value of £1 per annum for any number of years, as given in the published tables:—

(*1*) *Formula,* $$P = \left(\frac{R^N - 1}{R^N r}\right)$$

by logs.: *Log.* (*Present value of £1 per annum*) = *Log.* $(R^N - 1)$ − *Log.* R^N − Log. r.

(*2*) *By Thoman's method:—*

Formula, $$P = \frac{1}{a^n}$$

by logs.: *Log.* (*Present value of £1 per annum*) = 10 − *Log.* a^n

B. To find the present value of any annuity for any number of years:—

(*1*) *Formula,* $$P = Ay\left(\frac{R^N - 1}{R^N r}\right)$$

by logs.: *Log.* (*Present value of annuity*) = *Log. Annuity* + *Log.* $(R^N - 1)$ − *Log.* R^N − *Log.* r

(*2*) *By Thoman's method:—*

Formula, $$P = \frac{Ay}{a^n}$$

by logs.: *Log.* (*Present value of annuity*) = *Log. annuity* + 10 − *Log.* a^n

The present chapter deals only with the formula $P = Ay\left(\frac{R^N - 1}{R^N r}\right)$. *Thoman's method and formulæ are fully described in Chapter IX.*

General Rules deduced from the above formulæ.

To find the present value of any annuity for any number of years. *Author's Standard Calculation Form, No. 4.*

Rule 1. If the rate per cent. be not given in Table IV or in Thoman's Tables:—

Proceed by the formula relating to Table IV.
Calculation (VII) 2 A.

Rule 2. If the rate per cent. be given in Table IV:—

Multiply the amount given in the table, by the given annuity. The product is the present value required. *Calculation (VII) 2 B.*

Rule 3. If the rate per cent. be given in Thoman's Table:—

To the log. of the given annuity add 10, and deduct therefrom the log. of a^n *as given by Thoman. The remainder is the log. of the present value required.*
Calculation (VII) 2 C.

To find the rate per cent., or number of years, proceed as shown in the standard form for the purpose, given in Chapter X.

Derivation of the Formula. In order to find the formula relating to the present value of an annuity due at the end of each year of a given term, the most direct method is to consider an annuity of £1 in order to demonstrate the principle involved and to arrive at the necessary modification in the previous formula relating to the amount of an annuity. It will be readily seen that the present value of an annuity for any number of years is the same as the present value of the sum to which that annuity will amount in the same period at the same rate per cent. It has been shown in Chapter VI that the amount of an annuity of £1 may be found by the formula:—

$$M = \frac{R^N - 1}{r}$$

and that the present value of £1 due at the end of any number

of years is found by the formula, relating to Table II and described in Chapter V, namely:—

$$P = \frac{A}{R^N}$$

but $A=1$, therefore $P = \frac{1}{R^N}$.

Consequently by multiplying these two formulæ together the required formula for the present purpose is obtained as follows:—

$$P = (\text{present value of £1 per annum}) = \frac{R^N - 1}{r} \times \frac{1}{R^N} = \frac{R^N - 1}{R^N\, r}$$

and the formula to find the present value of any annuity, Ay, for any number of years becomes:—

$$P = Ay\left(\frac{R^N - 1}{R^N \cdot r}\right)$$

There is a similarity between the formulæ relating to Tables III and IV, namely, that they are both based upon the factor $\frac{R^N - 1}{r}$. In both cases, as will be seen by an inspection of the standard calculation forms, Rule 1, the method consists in adding to the log. of the annuity the log. of $R^N - 1$, and deducting from the sum of the logs. the log. of r. This gives the desired result in the case of Table III relating to the amount of an annuity, but in Table IV relating to the present value of an annuity, the log. of R^N is previously deducted from the log. of the annuity, which is equivalent to saying that the present values in Table IV may be found by dividing the amounts in Table III by R^N; but the values of $\frac{1}{R^N}$ are given in Table II, therefore the amounts in Table IV are equal to the amounts in Table III, multiplied by the amounts in Table II, or divided by the amounts in Table I.

Calculations. Having found the above formula relating to the present value of an annuity of £1 for any number of years, two calculations will now be made by its aid upon the author's standard calculation form, No. 4. Both cases will include the three methods of which the general rules are stated at the head of this chapter, namely, by formula, by the published tables, and by Thoman's method and tables. The first calculation will deal only with an annuity of £1, and will show the method of computing the amounts given in Table IV. Calculation (VII) 1.

The second calculation will deal with an annuity of stated amount, and will illustrate the method to be adopted in actual practice. Calculation (VII) 2.

Calculation (VII) 1.

Standard Calculation Form, No. 4.

To find the present value of an annuity for any number of years, and thereby prove the accuracy of the published table.

Table IV.

Required the present value of £1 per annum for 10 years at 5 per cent. per annum, compound interest.

(A) By Formula. $P = Ay\left(\frac{R^N - 1}{R^N r}\right)$ Rule 1, Chapter VII.

Log $R^N - 1$: Log. Ratio	R	1·05	0·0211893
multiply Log. R by	N	10	
	R^N	$(1·05)^{10}$	0·2118930
Convert Log. to ordinary number	R^N	1·6289	
deduct unity	-1	1·	
Log. of this is	$R^N - 1$	0·6289	$\bar{1}$·7985779
Log. Annuity	Ay	1·	0·0000000
add Log. $(R^N - 1)$ above	$R^N - 1$		$\bar{1}$·7985779
			$\bar{1}$·7985779
deduct Log. R^N above	R^N		0·2118930
			$\bar{1}$·5866849
deduct Log. r	r		$\bar{2}$·6989700
	P		0·8877149

Required present value, £7·72174.

(B) By Table IV. $P = Ay\left(\frac{R^N - 1}{R^N r}\right)$ Rule 2, Chapter VII.

Table IV. 10 years, 5 per cent. Present Value £1 per annum	$\frac{R^N - 1}{R^N r}$	7·72174
add Log. Annuity	Ay	
	P	

Required present value, £7·72174. This amount is given in Table IV.

(C) By Thoman's Table $P = \frac{Ay}{a^n}$ Rule 3, Chapter VII.

5 per cent. 10 years.

Log. Annuity	Ay	1·	0·
add 10			10·0000000
deduct Log.	a^n		9·1122851

Calculation (VII) 2.

Standard Calculation Form, No. 4.

To find the present value of an annuity for any number of years. Table IV.

Required the present value of £500 per annum for 10 years at 5 per cent. per annum, compound interest.

(A) By Formula. $P = Ay \left(\frac{R^N - 1}{R^N r}\right)$ Rule 1, Chapter VII.

Log $R^N - 1$: Log. Ratio *multiply* Log. R by	R N	1·05 10	0·0211893 10
	R^N	$(1·05)^{10}$	0·2118930
Convert Log. to ordinary number *deduct* unity	R^N -1	1·6289 1·	
Log. of this is	$R^N - 1$	0·6289	$\bar{1}$·7985779
Log. Annuity *add* Log. $(R^N - 1)$ above	Ay $R^N - 1$	500	2·6989700 $\bar{1}$·7985779
deduct Log. R^N above	R^N		2·4975479 0·2118930
deduct Log. r	r		2·2856549 $\bar{2}$·6989700
	P		3·5866849

Required present value, £3860·867.

(B) By Table IV. $P = Ay \left(\frac{R^N - 1}{R^N r}\right)$ Rule 2, Chapter VII.

Table IV. 10 years, 5 per cent. Present Value £1 per annum	$\frac{R^N - 1}{R^N r}$	7·72174	0·8877149
Add Log. Annuity	Ay	500	2·6989700
	P		3·5866849

Required present value, £3860·867.

(C) By Thoman's Table. $P = \frac{Ay}{a^n}$ Rule 3, Chapter VII.

5 per cent. 10 years.

Log. Annuity	Ay	500	2·6989700
add 10 *deduct* Log.	 a^n		12·6989700 9·1122851
	P		3·5866849

Required present value, £3860·867.

CHAPTER VIII.

COMPOUND INTEREST AS APPLIED TO AN ANNUAL OR OTHER PERIODIC PAYMENT (*Continued*).

TABLE V. The annuity which £1 will purchase for any number of years, or of which £1 is the present value.

$$Ay = P\left(\frac{R^N r}{R^N - 1}\right)$$

FORMULÆ AND RULES DEDUCED THEREFROM. GENERAL REMARKS AS TO TABLE V AND ITS RELATION TO AN EQUAL ANNUAL INSTALMENT, OF PRINCIPAL AND INTEREST COMBINED. THIS TABLE GIVES THE ACTUAL VALUES OF THOMAN'S LOG. FACTOR, a^n. DERIVATION OF FORMULA AND APPLICATION TO COMPILATION OF TABLES AND TO CALCULATIONS. CALCULATIONS TO DEMONSTRATE THE THEORETICAL CONCLUSIONS BOTH AS REGARDS THE PUBLISHED TABLES AND PRACTICAL EXAMPLES. THOMAN'S METHOD AND FORMULA.

AUTHOR'S STANDARD CALCULATION FORM, No. 5.

Formulæ.

A. *To find the annuity which £1 will purchase for any number of years, or of which £1 is the present value, as given in the published tables*:—

(1) *Formula*, $Ay = \left(\frac{R^N r}{R^N - 1}\right)$

by logs.: *Log.* (*Annuity £1 will purchase*) $=$ *Log.* R^N $+$ *Log.* $r -$ *Log.* $(R^N - 1)$

(2) *By Thoman's method*:—

Formula, $Ay = a^n$

by logs.: *Log.* (*Annuity £1 will purchase*) $=$ *Log.* $a^n - 10$

B. To find the annuity which may be purchased with any given sum for any number of years:—

(*1*) *Formula,* $$Ay = P\left(\frac{R^N r}{R^N - 1}\right)$$

by logs.: Log. (required annuity) = Log. (principal sum) + Log. R^N + *Log.* r − *Log.* $(R^N - 1)$

(*2*) *By Thoman's method:—*

Formula, $$Ay = P\ a^n$$

by logs.: Log. (required annuity) = Log. (principal sum) + Log. $a^n - 10$

The present chapter deals only with the formula

$$Ay = P\left(\frac{R^N r}{R^N - 1}\right)$$

Thoman's method and formulæ are fully described in Chapter IX.

General Rules deduced from the above formulæ.

To find the annuity which may be purchased with any given sum for any number of years.

Author's Standard Calculation Form, No. 5.

Rule 1. If the rate per cent. be not given in Table V or in Thoman's Tables:—

Proceed by the formula relating to Table V.

Calculation (VIII) 2 A.

Rule 2. If the rate per cent. be given in Table V:—
Multiply the annuity given in the table, by the given sum. The product is the required annuity which may be purchased. *Calculation (VIII) 2 B.*

Rule 3. If the rate per cent. be given in Thoman's Tables:—
To the log. of the given sum, add the log. of a^n *as given by Thoman. Deduct 10 from the sum of the two logs. The remainder is the log. of the required annuity which may be purchased.*

Calculation (VIII) 2 C.

To find the rate per cent., or number of years, proceed as shown in the standard form for the purpose, given in Chapter X.

Tables III and IV, containing the amounts and present values of £1 per annum, correspond to Tables I and II relating to the amounts and present values of £1. There is a further

Table, No. V, given in Inwood and other published tables, which is useful in order to ascertain the annuity which may be purchased with a given sum of money, because anyone contemplating the purchase of an annuity generally has a definite sum to invest in this manner. Consequently it is required to know the annuity which £1 will purchase for any number of years, and from this can be ascertained by simple multiplication the annuity which any given sum will purchase.

But the principal value of this table lies in the fact that the amounts there given represent the respective annuities of which £1 is the present value. The importance of this will be recognised when it is remembered that this is the principle underlying the repayment of debt by an equal annual instalment of principal and interest combined, as laid down in Section 234 (4) of the Public Health Act, 1875. Table V represents Thoman's factor (a^n), and is a connecting link between £1 and £1 per annum considered both in regard to future amount and present value. By its aid the cumbersome factor $R^N - 1$, previously referred to, may be avoided in cases where the rate per cent. is included in Thoman's tables.

Derivation of the Formula. The formula relating to this table may be found by simple proportion without resorting to any algebraical calculation. It has been ascertained by Calculation (VII) 1, that £7·7217 is the present value of an annuity of £1, and such values are given in Table IV. It is required to find the annuity of which £1 is the present value.

It is obvious that it will be $\frac{1}{7{\cdot}7217}$ of £1.

Consequently, by dividing unity by the present value of an annuity of £1, as given in Table IV, the result is the annuity which may be purchased by £1, and may be expressed by the following rule:—

To find the annuity which £1 will purchase for any number of years, first ascertain by Table IV the present value of an annuity of £1 for the same period at the same rate per cent.; and divide 1 by the present value so found. The quotient will be the annuity which may be purchased by £1.

This rule simply means that to ascertain the annuity which £1 will purchase, unity is divided by the values given in

Table IV, but if it be reduced to terms of the annuity formula it becomes:—

$$\left.\begin{array}{l}\text{The annuity of which £1 is the present}\\ \quad\text{value, as given in Table V} \ldots \quad \ldots\end{array}\right\} = \frac{1}{\dfrac{R^N - 1}{R^N r}} \text{ or } \frac{R^N r}{R^N - 1}$$

Table V. Calculations. Having found the above formula relating to the annuity which £1 will purchase for any number of years or the annuity of which £1 is the present value, two calculations will now be made by its aid upon the author's standard form, No. 5. Both cases will include the three methods of which the general rules are stated at the head of this chapter, namely, by formula, by the published tables, and by Thoman's method and tables. The first calculation will deal only with the annuity which £1 will purchase, or of which £1 is the present value, and will show the method of computing the amounts given in Table V. Calculation (VIII) 1.

The second calculation will deal with a stated amount to be invested in an annuity, or of which it is required to ascertain the future equivalent expressed in an annual payment, and will illustrate the method to be adopted in actual practice Calculation (VIII) 2.

Calculation (VIII) 1.

Standard Calculation Form, No. 5.

To find the annuity which a present sum will purchase for any number of years, and also the equal annual instalment of principal and interest combined, and thereby to prove the accuracy of the published table. Table V.

Required the annuity which £1 will purchase for 10 years at 5 per cent. per annum, compound interest.

(A) By Formula. $Ay=P\left(\frac{R^N r}{R^N-1}\right)$ Rule 1, Chapter VIII.

Log R^N-1	Log. Ratio	R	1·05	0·0211893
	multiply Log. R by	N	10	10
		R^N	$(1·05)^{10}$	0·2118930
	Convert Log. to ordinary number	R^N	1·6289	
	deduct unity	-1	1·	
		R^N-1	0·6289	$\bar{1}$·7985779
	Log. Present Sum	P	1·	0·
	add Log. R^N above	R^N	1·6289	0·2118930
	Log. r	r	0·05	$\bar{2}$·6989700
				$\bar{2}$·9108630
	deduct Log. (R^N-1) above	R^N-1		$\bar{1}$·7985779
		Ay		$\bar{1}$·1122851

Required annuity, £0·129546.

(B) By Table V. $Ay=P\left(\frac{R^N r}{R^N-1}\right)$ Rule 2, Chapter VIII.

Table V. 10 years, 5 per cent. Annuity £1 will purchase	$\frac{R^N r}{R^N-1}$	0·1295
add Log. Present Sum	P	
	Ay	

Required annuity, £0·1295. This amount is given in Table V.

(C) By Thoman's Table. $Ay=P\,a^n$ Rule 3, Chapter VIII.
5 per cent. 10 years.

Log. Present Sum	P	1·	0·
add Log. a^n	a^n		9·1122851
			9·1122851
deduct 10	Ay		$\bar{1}$·1122851

Required annuity, £0·129546.

Calculation (VIII) 2.

Standard Calculation Form, No. 5.

To find the annuity which a present sum will purchase for any number of years. Table V.

To find the annuity which may be purchased with £6288·94 for 10 years at 5 cent. per annum, compound interest.

(A) By Formula $Ay = P\left(\frac{R^N r}{R^N - 1}\right)$ Rule 1, Chapter VIII.

Log $R^N - 1$: Log. Ratio	R	1·05	0·0211893
multiply Log. R by	N	10	10
	R^N	$(1·05)^{10}$	0·2118930
Convert Log. to ordinary number	R^N	1·6289	
deduct unity	−1	1·	
	$R^N - 1$	0·6289	$\bar{1}$·7985779
Log. Present Sum	P	6288·94	3·7985779
add Log. R^N above	R^N	1·6289	0·2118930
Log. r	r	0·05	$\bar{2}$·6989700
			2·7094409
deduct Log. $(R^N - 1)$ above	$R^N - 1$	0·6289	$\bar{1}$·7985779
	Ay		2·9108630

Required annuity, £814·447.

(B) By Table V. $Ay = P\left(\frac{R^N r}{R^N - 1}\right)$ Rule 2, Chapter VIII.

Table V. 10 years, 5 per cent. Annuity £1 will purchase	$\frac{R^N r}{R^N - 1}$	0·1295	$\bar{1}$·1122851
add Log. Present Sum	P	6288·94	3·7985779
	Ay		2·9108630

Required annuity, £814·447.

(C) By Thoman's Table. 5 per cent. 10 years. $Ay = P\,a^n$ Rule 3, Chapter VIII.

Log. Present Sum	P	6288·94	3·7985779
add Log. a^n	a^n		9·1122851
			12·9108630
deduct 10	Ay		2·9108630

Required annuity, £814·447.

CHAPTER IX.

THOMAN'S LOGARITHMIC TABLES OF COMPOUND INTEREST AND ANNUITIES.

EXPLANATION OF THOMAN'S SYMBOLS, R^N, AND, a^n, AND THEIR RELATION, SEPARATELY OR IN COMBINATION, TO THE FORMULÆ ALREADY ASCERTAINED. THOMAN'S METHOD OF STATING LOG. OF a^n BY ADDING 10 TO THE LOG.

AUTHOR'S STANDARD CALCULATION FORMS, 1 TO 5.

Symbols used by Thoman :—

R^N = *the amount of £1 in any number of years.*

a^n = *the annuity which £1 will purchase for any number of years.*

$$a^n = \frac{R^N r}{R^N - 1} \qquad \textit{(Table V.)}$$

Comparison with previous formulæ :—

Chapter.	Table No	Giving Values of	General Formulæ	Thoman's Logarithmic Formulæ.
IV.	*I.*	*Amount of £1*	R^N	R^N
V.	*II.*	*Present value of £1*	$\frac{1}{R^N}$	$\frac{1}{R^N}$
VI.	*III.*	*Amount of £1 per annum*	$\frac{R^N - 1}{r}$	$\frac{R^N}{a^n}$
VII.	*IV.*	*Present value of £1 per annum* ...	$\frac{R^N - 1}{R^N r}$	$\frac{1}{a^n}$
VIII.	*V.*	*Annuity which £1 will purchase*	$\frac{R^N r}{R^N - 1}$	a^n

The logarithmic equivalents of the above formulæ and the rules based thereon are given at the head of the chapter dealing with each table.

Included in Inwood, and in other published tables of compound interest, are a series of valuable tables by M. Fédor Thoman, of the Soc. Crédit Mobilier, of Paris, the author of "Logarithmic Tables of Interest, etc." These tables are of great assistance in the solution of many problems; and the whole of the formulæ already obtained by derivation from the algebraical formula for a geometrical progression will now be compared with the simplified formulæ given by Thoman. A glance at the above comparative table will show that the whole of the formulæ already obtained by derivation from the formula $A = P\ R^N$, may be expressed by some modification of the factors R^N, and r, which have already been fully explained in previous chapters. The table at the end of Chapter V contains the values of R, and r, for many rates per cent. likely to be required in practice, and also gives the corresponding logs. of these values; and it has been explained how to find by means of the formulæ the values and logs. of R, and r, for any intermediate rates per cent. not included in the above table. The values of R^N may be obtained by multiplying the log. of R by the number of years, as shown in Calculation (IV) 2 and others.

Problems involving an annual or other periodic payment, as in Tables III, IV, and V, cause the introduction of a variation of R^N, namely $(R^N - 1)$, which imports a new calculation which, although not of itself difficult, is inconvenient because it is necessary to convert the log. of R^N, to an ordinary number before deducting unity, and afterwards to find the log. of the remainder. This might be avoided by preparing tables of $(R^N - 1)$, and the corresponding logs., for each rate per cent. for any number of years.

In the above formulæ relating to an annual sum in Tables III, IV, and V, the factor $(R^N - 1)$ is always associated with r, or $(R^N r)$, and R^N is the factor relating to the amount and present value of £1 as shown by Tables I and II. By combining the factors $(R^N - 1)$, and R^N, a connecting link is obtained between Tables I and II relating to £1, and Tables III, IV, and V, relating to the amount and present value of £1 per annum; and this is the principle underlying Thoman's method.

They are merely tables, and do not enunciate any new principle, but by giving under each rate per cent. for various numbers of years the logs. of two factors only, they enable any calculation to be made, at the rates included in the tables, without any further reference to the ordinary published tables

I to V. Thoman's tables have two advantages over the ordinary tables of compound interest in that (1) they are worked out for fractional eighths per cent. up to 6 per cent., and (2) they give the logs. direct and thereby avoid any reference to the log. tables; but since they are worked out for a limited number only of rates per cent. they do not dispense entirely with the method of calculation by means of the formulæ previously stated, and these methods will therefore be included in subsequent chapters as well as Thoman's method.

The factors included in Thoman's tables are R^N and a^n. R^N is the factor governing Tables I and II, without any alteration, and Thoman's tables may be referred to instead of finding the logs. of these values by the methods shown in the various calculations, using the ordinary tables and logs.

a^n is used by Thoman to denote the annuity which £1 will purchase or of which £1 is the present value.

The logs. of a^n in Thoman's tables are, purely for convenience of calculation and perhaps for facilitating printing, given in a different form to the logs. of R^N in the same tables. In Table I (R^N) the values are all greater than unity, hence the characteristics of the logs. of these values are always positive. In Table V the values are all decimals of unity, and the logs. of these values have negative characteristics. Thoman adds 10 to the characteristics of the logs. of the values of a^n in Table V, and thus, bearing in mind that any calculation can be made by means of R^N and a^n, it is possible to eliminate the troublesome negative characteristic altogether. All that is required is to correct the characteristic of the final log. by adding 10 in the case of Tables III and IV, and deducting 10 in the case of Table V, to or from the resulting log. before ascertaining the antilog., or numerical equivalent. Thoman's logs. of a^n may be treated in the ordinary way, by using the mantissa given in the table and converting the characteristic there given to the proper minus quantity, *i.e.*, 10 minus the given characteristic. For the sake of clearness the method of deducting or adding 10 from or to the log. has been adopted in the whole of the standard calculation forms prepared by the author.

There are two methods of connecting the ordinary tables and formulæ relating to £1, and £1 per annum, namely, either by means of Table I and III, dealing with the respective amounts (as adopted by Thoman to derive the factor a^n as shown by Table V) or by means of Tables II and III, leading to Table IV, which is the reciprocal of Table V, as shown when dealing with the latter table in Chapter VIII.

Thoman's method will now be applied in order to derive Table V, or the formula relating thereto, from:—

Table I, the amount of £1 for any number of years,

and, *Table III*, the amount of £1 per annum in any number of years,

taking, in each case, a period of 10 years, and a rate of accumulation of 5 per cent. per annum, as follows:—

Table I, amount of £1 ... ... 1·6289 or R^N

Table III, amount of £1 per annum ... 12·5779 or $\frac{R^N - 1}{r}$

The annuity which £1 will purchase is obtained by dividing 1·6289 by 12·5779, and the formula corresponding thereto may also be obtained by dividing the corresponding formulæ as follows:—

$$\text{Table V} = \frac{\text{Table I}}{\text{Table III}} = \frac{1·6289}{12·5779} = \frac{R^N}{\frac{R^N - 1}{r}}.$$

Stated in actual values:—

$$\text{Table V} = \frac{1·6289}{12·5779} = 0·1295$$

as may be found by actual calculation, or obtained by direct reference to Table V.

Stated in terms of the above formulæ:—

$$\text{Table V} = \frac{R^N}{\frac{R^N - 1}{r}} \text{ or } \frac{R^N r}{R^N - 1}$$

which is the formula relating to Table V, as shown in Chapter VIII, giving the annuity which £1 will purchase for any number of years. But Thoman's symbol a^n, although expressed in logarithmic form, represents the same factor; therefore the formula relating to Table V, as derived in Chapter VIII, from the simple formula $A = P\,R^N$, may be replaced by Thoman's symbol a^n, with the result that:—

$$\text{Table V} = \frac{R^N r}{R^N - 1} = a^n \text{ of Thoman.}$$

The above formula, relating to Table V, contains the three factors, R^N, r, and $(R^N - 1)$, already referred to and fully explained in previous chapters, in order to express the relation between the respective amounts and present values of £1, and

of £1 per annum. Thoman, by adopting the symbol a^n, eliminates the factor (R^N-1) altogether.

The factor (R^N-1) may, however, be ascertained from Thoman's tables if required as follows:—

$$(R^N-1)=\frac{R^N r}{a^n}$$

or by logarithms:

$$\text{Log.}\ (R^N-1)=\text{Log.}\ R^N+\text{Log.}\ r-\text{Log.}\ a^n.$$

and Thoman's a^n may be found from the above factors, as follows:—

$$\text{Log.}\ a^n=\text{Log.}\ R^N+\text{Log.}\ r-\text{Log.}\ (R^N-1).$$

The whole of the formulæ previously ascertained by derivation from the simple formula $A=P\ R^N$, may now be expressed in terms of Thoman's factors of R^N, and a^n, as follows:—

Table I. R^N *The amount of £1:—*

will be expressed by Thoman's factor R^N

Table II. $\frac{1}{R^N}$ *The present value of £1:—*

will be expressed by Thoman's factor $\frac{1}{R^N}$

Table III. $\left(\frac{R^N-1}{r}\right)$ *The amount of £1 per annum:—*

It has been proved that:—

$$\text{Table V}=\frac{\text{Table I}}{\text{Table III}}$$

And transposed, it will be seen that:—

$$\text{Table III}=\frac{\text{Table I}}{\text{Table V}}$$

but Table I $=$ R^N

and Table V $=$ a^n

Therefore Table III will be expressed

by Thoman's symbols $\frac{R^N}{a^n}$

Table IV. $\left(\frac{R^N - 1}{R^N r}\right)$ *The present value of £1 per annum:—*

It has been proved in Chapter VIII, that Table IV, is the reciprocal of Table V; and it has been shown above, that Table V, is expressed by Thoman's symbol, a^n.

Therefore Table IV will be expressed by Thoman's factor $\frac{1}{a^n}$

Table V. $\left(\frac{R^N r}{R^N - 1}\right)$ *The annuity which £1 will purchase:—*

This, as shown above, is the equivalent of Thoman's factor a^n

The above formulæ by Thoman may be stated logarithmically as follows, using the logs. of a^n increased by 10, as given in Thoman's tables:—

$$\text{Table I} = R^N = \text{Log } R^N$$

$$\text{Table II} = \frac{I}{R^N} = \text{Log } 1 = 0 - \text{Log } R^N$$

$$\text{Table III} = \frac{R^N}{a^n} = \text{Log } R^N - \text{Log } a^n \ (\textit{add } 10)$$

$$\text{Table IV} = \frac{1}{a^n} = \text{Log } 1 = 0 - \text{Log } a^n \ (\textit{add } 10)$$

$$\text{Table V} = a^n = \text{Log } a^n \ (\textit{deduct } 10)$$

and any problem may be solved by adding to, or deducting from, the log. of the given sum or annuity the logs. of R^N and a^n as given by Thoman, as shown in the various standard calculation forms prepared by the author given in Chapter X.

In using the above log. formulæ it is important to bear in mind the previous remarks as to the addition of 10 to the log. of a^n in the tables by Thoman, and the following examples will make the matter clear. It affects only Tables III, IV and V.

Taking the figures relating to a period of 10 years and a rate of accumulation of 5 per cent. in each case, the above log. formulæ will now be applied to find the amounts given in those tables, which, of course, relate to £1 only:—

Table III. Required the amount of £1 per annum for 10 years at 5 per cent.

Calculation (VI) 1.

Log. $R^N + 10$ ...	$= 10{\cdot}211\ 8930$
deduct Log. a^n	$= 9{\cdot}112\ 2851$
	$1{\cdot}099\ 6079$
which is the Log. of	£12·5779

Table IV. Required the present value of £1 per annum for 10 years at 5 per cent.

Calculation (VII) 1.

Log. $1 = 0 + 10$...	$= 10{\cdot}000\ 0000$
deduct Log. a^n	$= 9{\cdot}112\ 2851$
	$0{\cdot}887\ 7149$
which is the Log. of	£7·7217

The above examples show, that in using the logs. of a^n as given in Thoman's tables in conjunction with the log. formulæ relating to Tables III and IV, the log. of a^n (which is increased by 10 in the tables), is deducted, and consequently 10 must be added to the resulting log., but it is immaterial where 10 is added. In the above example 10 has been added to the log. of R^N as given in Thoman's tables.

In the case of Table V, the factor a^n represents the values given in the table, and as Thoman's logs. are increased by 10, it only remains to deduct 10 therefrom in order to find the true log. of the annuity required.

In previous chapters dealing with Tables I to V and in the following chapters dealing with other calculations, the formulæ and rules relating to the method by Thoman will be found at the head of each chapter, without any further explanation.

CHAPTER X.

STANDARD CALCULATION FORMS, PREPARED BY THE AUTHOR.

THE FIVE PUBLISHED TABLES OF COMPOUND INTEREST. THE THREE METHODS OF CALCULATION IN EACH FORM AND THE CORRESPONDING RULES RELATING THERETO. MEANING OF ALL SYMBOLS AND FACTORS, AND THE VARIOUS METHODS OF FINDING THE ACTUAL LOG. VALUES. SIX STANDARD FORMS, WITH REFERENCES TO THE ABOVE AND LIST OF PROBLEMS TO WHICH EACH MAY BE APPLIED. ALSO STANDARD CALCULATION FORMS TO FIND THE EXACT OR APPROXIMATE RATE PER CENT. OR NUMBER OF YEARS, IN CONNECTION WITH EACH OF THE FIVE TABLES OF COMPOUND INTEREST AND THE SINKING FUND INSTALMENT.

THE FIVE STANDARD TABLES OF COMPOUND INTEREST RELATING TO THE AMOUNT AND PRESENT VALUE OF £1, AND OF £1 PER ANNUM.

Table I. The amount of £1 in any number of years.
Chapter IV, Form I.

Table II. The present value of £1 due at the end of any number of years. Chapter V, Form II.

Table III. The amount of £1 per annum in any number of years.
Chapters VI and XIII, Forms III and IIIx.

Table IV. The present value of £1 per annum for any number of years. Chapter VII, Form IV.

Table V. The annuity which £1 will purchase for any number of years. Chapter VIII, Form V.

In these tables the actual values only are generally given, and not their logarithmic equivalents which are found in the tables of M. Fédor Thoman.

method by formula will be found invaluable, inasmuch as in many cases the rate per cent. per period will probably not be found worked out in any of the published tables.

THE THREE METHODS OF CALCULATION IN EACH FORM.

A. *By formula—Rule 1.* The method of making the calculations in this manner is fully explained in the following notes and in the standard forms.

B. *By the published tables of compound interest—Rule 2.* The first step is to ascertain from the tables the actual values relating to £1 under similar conditions as to period and rate per cent. This amount, multiplied by or divided into the sum in respect of which the calculation is to be made, gives the result required.

C. *By Thoman's Tables—Rule 3.* This method is fully described in the standard forms, and consists merely of various combinations of the logs. of R^N and a^n and of the amounts in the problem. Thoman's tables are fully described in Chapter IX. If the results be required to be correct to the utmost decimal point the method by Thoman should be adopted because these tables give the actual log. values.

A full explanation of the rules in each form is given at the head of the chapter dealing with the subject matter of each form.

THE RATE PER CENT. AND THE NUMBER OF YEARS. In addition to the six standard forms, the author has prepared ten forms showing the methods of determining the rate per cent. and the number of years in connection with each of the five tables of compound interest and the sinking fund instalment. These latter forms each contain particulars of an example, worked out in full in the book and to which a reference is made.

It will be noticed that the results obtained are in several cases approximate only, especially as regards the rate per cent. as expressed by the factors R and r, which, however, can be determined to any required degree of accuracy only by methods which are far too technical to be included in a work of this nature. For all practical purposes an approximation of the rate per cent. is sufficient, and this may generally be obtained

from the published tables of compound interest giving either the actual values or their logarithmic equivalents. This difficulty in finding the exact rate per cent. arises only in the case of an annuity or other periodic payment. In the case of the amount and present value of a sum of money [Tables I and II] the calculation is a simple one depending upon the value of R^N. If the value of this be known as well as one of the factors, the other may be found as shown in the following standard forms for finding the rate per cent. and number of years. The factor relating to all annuities is $\left(\frac{R^N-1}{r}\right)$ and represents the amount of £1 per annum at the end of N years. The rate per cent. cannot be determined exactly from this factor, and the method by approximation is too long. The practical method of finding the approximate rate per cent. of accumulation of an annuity is to reduce the actual example to terms of an annuity of £1 as given in the various published tables of compound interest. Having done so, a reference is made to the tables and the nearest figure ascertained. This figure is adopted in the calculation, and if necessary the resulting error is calculated and corrected. An example of this may be found on referring to Chapter XXXII, where the effect of taking the equated period at 23, instead of 23·136 years, is fully explained and accounted for.

Other Methods of Calculation. In addition to the problems which may be solved by means of the standard forms included in this chapter there are several others which occur in the course of the book, and which are fully explained as they arise. In all cases the calculation has been made in such a form that it may be adapted to any similar problem, and it is not necessary to repeat the forms here. The following list includes the whole of these special calculations:—

(1) A stated sum is required to be set aside and accumulated at compound interest for a stated number of years. At the end of that time the annual sum ceases, but the amount then in the fund continues to accumulate for a further stated term. The rate of accumulation may be varied or not at the end of the first period. It is required to ascertain the amount in the fund at the end of the second period. Statement XVI. D. 1.

(2) A stated amount is required to be provided at the end of a prescribed period by the accumulation of an annual sum

to be set aside during the early years of such period only. The amount then in the fund will continue to accumulate until the end of the prescribed period. It is required to ascertain the annual amount to be set aside during the first part of the prescribed period.

Calculation (XXXIV) G.

(3) A stated annual sum to be set aside for a prescribed period and accumulated at a definite rate per cent., is sufficient to provide a stated amount at the end of that period. Before the expiration of the period a change is made in the conditions affecting either the period or the rate per cent., or both, but not the amount to be provided. It is required to ascertain by one calculation the equivalent future annual sum under the amended conditions.

Statement XXVI. D.

Symbols. The following is a complete list of the symbols used in the various formulæ, and in the standard calculation forms, with the meaning of each symbol, all of which have been fully described in Chapter III:—

A denotes the amount or ultimate sum to which the present sum P will accumulate in N years at the ratio or common factor R. It represents both the ultimate sum, to which a stated present sum will amount at the end of the period, as well as the stated sum, due at the end of the period, of which the present value P is required.

P denotes the principal sum in hand; and represents also the present value of a definite sum of money A due at the end of a stated period of years; it denotes also the present value of an annuity or other periodic sum, Ay, payable at the end of each of a stated number of years or periods N.

R denotes the ratio or common factor existing between each term of the progression, or the amounts of £1 at the end of each succeeding year. It is in all cases £1 increased by interest upon £1 for one year at the rate per cent. in question. It corresponds with the algebraical factor r.

r denotes the interest upon £1 for one year or period at the stated rate per cent. It is always less than the above factor R, by unity.

$$R-1=r.$$

The actual rate per cent. is never used in calculations involving compound interest, but is always expressed in

its relation to £1 only, as R, and r, above. This term r is *not* the equivalent of the algebraical factor r, in a geometrical progression.

N denotes the number of years, or other equal periods, and must not be confounded with the factor n in the algebraical formula for a geometrical progression which represents the number of terms in the progression. For this reason it is expressed by a capital letter. This term is the equivalent of the algebraical term $n-1$.

Ay denotes the annuity or other periodic sum to be paid, set aside or received at the end of each year or period, N.

M denotes the sum to which the annuity or other periodic sum Ay will amount, if accumulated for a stated number of years or periods, N, at a stated rate per cent.

FORMULÆ. The above symbols are combined in various ways in the formulæ given in the book resulting in various factors, and the following list has been prepared in order to show the meaning of such factors, and also the manner in which they may be found, not only in actual values, but also in their logarithmic equivalents.

The numbers in brackets in the author's standard calculation forms in this chapter refer to the following notes:—

Note	Symbol or factor	Remarks
(1)	R = ratio.	This may be found by adding to £1, interest upon £1 for one year; and the log. of R may be found from the log. tables or from the special table of those logs. given in Chapter V.
(2)	r = interest upon £1 for one year.	This may be found by deducting unity from the value of R above, and the log. of r may be found, as in note (1).
(3)	R^N.	This symbol corresponds with the symbol r^n of Thoman. The log. of R^N is found by multiplying the log. of R by the number of years or periods. The logs. of R^N are given in Thoman's tables. The actual values of R^N are given in Table I.

Note	Symbol or factor	Remarks
(3a)	$\frac{1}{R^N}$.	This factor will very rarely, if ever, be required to be found by calculation. The actual values are given in Table II. The log. of this factor may be found by deducting the log. of R^N from 0.
(4)	$R^N - 1$.	The actual values of this factor are not given in Inwood or other published tables, although they may be found by deducting unity from the values given in Table I. The log. of this factor is not given in Thoman's tables, but may be found by converting the log. of R^N there stated into its equivalent ordinary number or anti-log., deducting unity therefrom, and finding the log. of the remainder. The log. of R^N-1 so found may be proved by Inwood, by deducting unity from the amount given in Table I, and finding the log. of the remainder. The log. of R^N-1 may be found by Thoman's tables as follows:— Log. $R^N + \log. r + 10 - \log. a^n$, log. r being found, as explained in note (2) above.
(5)	$\frac{R^N - 1.}{r}$	The actual values of this factor are given in Table III in Inwood. The logs. of this factor may be found by Thoman's tables as follows:— Log. $R^N + 10 - \log. a^n$.
(6)	$\frac{R^N - 1.}{R^N r.}$	The actual values of this factor are given in Table IV in Inwood or other similar tables. The logs. of this factor may be found from Thoman's tables by deducting the log. of a^n, there given, from 10.

Note	Symbol or factor	Remarks
(7)	$\dfrac{R^N r}{R^N - 1}$.	The actual values of this factor are given in Table V in Inwood or other similar tables. The logs. of this factor may be found from Thoman's tables by deducting 10 from the log. of a^n there given.
(8)	a^n.	This is a term employed by Thoman to denote the annuity, £a per annum, which £1 will purchase for n years; and the actual values of which are given in Table V. The logs. given in Thoman's table are as explained in Chapter IX, the true logs. of a^n increased by 10. The relations between a^n and the above symbols are explained briefly in the foregoing notes and fully in Chapter IX. This factor is extremely useful for finding the equal annual instalment of principal and interest combined (the annuity method, Chapter XII).

Standard Calculation Form, No. 1.

TABLE I. To find the future amount of a present sum.

Chapter IV.

This form has been used in the solution of problems of the following nature:—

	Calculation.
To find the amount of loan which will be provided by the future accumulation of the present investments representing a sinking fund ...	(XV) 4.
To find the amount of loan which will be unprovided for if an ascertained deficiency in a sinking fund remains uncorrected	(XV) 6.
To find the amount of loan which will be provided by the future accumulation of the proceeds of sale of assets paid into the fund	(XVII) 3.

Standard Calculation Form, No. 1.

TABLE I. To find the future amount of a present sum.
The following rules are explained at the head of Chapter IV.
Here state the general nature of the problem. Calculation No.

Here state full details of the actual problem.

(A) By Formula. $A = P R^N$ Rule 1.

		Values.	Logs.
Log. R^N	Log. Ratio (1)	R	
	Multiply Log. R by	N	
	(3)	R^N	
	Log. Present Sum	P	
	add Log. R^N above (3)	R^N	
		A	

Required future amount, £

(B) By Table I. $A = P R^N$ Rule 2.

Table I. years per cent.		
Amount of £1 (3)	R^N	
add Log. Present Sum	P	
	A	

Required future amount, £

(C) By Thoman's Table. $A = P R^N$ Rule 3.
per cent. years

Log. Present Sum	P	
add Log. R^N (3)	R^N	
	A	

Required future amount, £

The Amount and Present Value of One Pound.

Tables I and II. Standard Forms, 1 and 2.

To find the number of years:

based on Calculation (XVI) 5.

Given factors:

Present sum	P	9463·00	
Amount thereof ...	A	11239·07	
Rate per cent. ...		$3\frac{1}{2}$	
Ratio	R	1·035	
Interest of £1 ...	r	0·035	

Details of Method:

find	Log. A	11239·07	4·0507305
find, and deduct ...	Log. P	9463·00	3·9760288
difference ...	Log. R^N		0·0747017
find	Log. R.	1·035	0·0149403

To find the number of years, divide the above log. of R^N by the above log. of R, as described in Chapter XXXII, and the quotient is the number of years required, in this case, 5 years.

The Amount and Present Value of One Pound.

Tables I and II. Standard Forms, 1 and 2.

To find the rate per cent:

based on Calculation (XV) 4.

Given factors:

Present sum	P	9463·00	
Amount thereof ...	A	14799·71	
Number of years ...	N	13	

Details of method:

find	Log. A	14799·71	4·1702533
find, and deduct ...	Log. P	9463·00	3·9760288
difference ...	Log. R^N		0·1942245
divide this log. by the number of years	N	13	0·0149403
which is the log. of ...	R	1·035	

To find the rate per cent., deduct unity from the above ratio and multiply the remainder by 100, or $3\frac{1}{2}$ per cent.

Standard Calculation Form, No. 2.

TABLE II. To find the present value of a sum due at a future date. Chapter V.

This form has been used in the solution of problems of the following nature:—

	Calculation.
To find the sum now payable which is the equivalent of a given loan payable at the end of a prescribed number of years. See Chapter XXXII.	
Given a stated sum, to find the accumulated amount of an annual instalment, to be set aside for a limited period only; the amount so found to accumulate for a further stated period, and then amount to the stated sum. "The method by step"	(XVI) 3.
The annual instalment is then found by means of standard form, No. 3x	(XVI) 4.

The methods of finding the rate per cent. and the number of years are similar to those given under Table I.

Standard Calculation Form, No. 2.

TABLE II. To find the present value of a sum due at a future date.

The following rules are explained at the head of Chapter V.

Here state the general nature of the problem. Calculation No.

Here state full details of the actual problem.

(A) By Formula. $P = \frac{A}{R^N}$ Rule 1.

			Values.	Logs.
Log R^N	Log. Ratio	(1)	R	
	Multiply Log. R by		N	
		(3)	R^N	
	Log. Future Sum		A	
	deduct Log. R^N above	(3)	R^N	
			P	

Required present value, £

(B) By Table II. $P = \frac{A}{R^N}$ Rule 2.

Table II. years per cent.		$\frac{1}{R^N}$	
Present value of £1	(3a)		
add Log. Future Sum		A	
		P	

Required present value, £

(C) By Thoman's Table. $P = \frac{A}{R^N}$ Rule 3.

per cent. years

Log. Future Sum		A	
deduct Log.	(3)	R^N	
		P	

Required present value, £

Standard Calculation Form, No. 3.

TABLE III. To find the amount of an annuity. Chapter VI.

This form has been used in the solution of problems of the following nature:—

	Calculation
To find the amount which should stand to the credit of a sinking fund at any time during the repayment period	(XV) 2.
To find the amount of loan which will be provided by the future accumulation of:—	
(1) the original annual sinking fund instalment	(XV) 5.
(2) the additional, augmented, or reduced annual sinking fund instalment	(XVI) 2.
(3) the income from the present investments representing the fund	(XIX) 1.
(4) the annual increment of the fund ...	(XIX) 4.

Standard Calculation Form, No. 3.

TABLE III. To find the amount of an annuity.

The following rules are explained at the head of Chapter VI.

Here state the general nature of the problem. Calculation No.

Here state full details of the actual problem.

(A) By Formula. $M = Ay \left(\frac{R^N - 1}{r}\right)$ Rule 1.

			Values.	*Logs.*
Log $R^N - 1$	Log. Ratio	(1)	R	
	Multiply Log. R by		N	
		(3)	R^N	
	Convert Log. to ordinary number		R^N	
	deduct unity		-1	1·
	Log. of this is	(4)	$R^N - 1$	
	Log. Annuity		Ay	
	add Log. $R^N - 1$ above	(4)	$R^N - 1$	
			$Ay\,(R^N - 1)$	
	deduct Log. r	(2)	r	
			M	

Required future amount, £

(B) By Table III. $M = Ay \left(\frac{R^N - 1}{r}\right)$ Rule 2.

Table III. years per cent. Amount of £1 per annum	(5)	$\frac{R^N - 1}{r}$
add Log. Annuity		Ay
		M

Required future amount, £

(C) By Thoman's Table. $M = Ay \left(\frac{R^N}{a^n}\right)$ Rule 3.

per cent. years

Log. Annuity		Ay
add Log. R^N in Table + 10	(3)	R^N
		$Ay\,R^N$
deduct Log. a^n	(8)	a^n
		M

Required future amount, £

The Amount of One Pound per Annum.

Table III. Standard Form 3.

To find the number of years:

based on Calculation (XVIII) 4.

Given factors:

Annuity	Ay	7500·00	
Amount of annuity	M	57468·48	
Rate per cent. ...		3·00	
Ratio	R	1·03	
Interest of £1 ...	r	0·03	

Details of Method:

find	Log. M	57468·48	4·7594297
find, and deduct ...	Log. Ay	7500·00	3·8750613
difference ...			0·8843684
find, and add... ...	Log. r		$\bar{2}$·4771213
the sum is ...	Log. R^N-1		$\bar{1}$·3614897
find value of this log.		0·22987	
add unity ...		1·	
which is the value of ...	R^N	1·22987	
find	Log. R^N		0·0898606
find	Log. R		0·0128372

To find the number of years, divide the above log. of R^N by the above log. of R, as described in Chapter XXXII, and the quotient is the number of years required, in this case, 7 years.

The Amount of One Pound per Annum.

Table III. Standard Form 3.

To find the rate per cent:

based on Calculation (XVIII) 7.

Given factors:

Amount of annuity	M	1176·58	
Annuity	Ay	58·3715	
Number of years ...		16	

Details of method:

find	Log. M	1176·58	3·0706241
find, and deduct ...	Log. Ay	58·3715	1·7662008
difference ...			1·3044233
find value of this log.		20·1569	

which is the amount of an annuity of one pound for 16 years at the required rate per cent.

To ascertain the rate per cent., refer to Table III, giving the amounts of one pound per annum, and find the nearest value to the above amount of 20·1569 in 16 years. If the rate so found is not near enough, refer to Thoman's tables and find the nearest log. to 1·3044233, which is ascertained by deducting the log. of a^n from the log. of R^N, plus 10.

Required rate per annum, 3 per cent.

Note. In cases where the rate per cent. is not included in the published tables of compound interest, or in Thoman's tables, the above method will give only approximate results.

Standard Calculation Form, No. 3x.

TABLE III. To find the annual sinking fund instalment.

Chapter XIII.

This form has been used in the solution of problems of the following nature:—

	Calculation
To find the annual sinking fund instalment, to be set aside out of revenue or rate, and accumulated at compound interest, to repay a stated loan at the end of a prescribed period	(XV) 1.
To find the annuity which will amount to a stated sum in any number of years	
To find the additional annual sinking fund instalment required to provide the amount of loan which will be unprovided for owing to a deficiency in the amount in the fund	(XVI) 1.
To find the amount by which the original annual sinking fund instalment may be reduced in consequence of the withdrawal, during the repayment period, of part of the loan from the operation of the fund	(XVIII) 1.
To find the future annual increment to be added to the fund, and accumulated at compound interest, to provide the balance of loan, which will not be provided by the future accumulation of the present investments representing the fund	(XVI) 9.

Standard Calculation Form, No. 3x.

TABLE III. To find the annual sinking fund instalment.

The following rules are explained at the head of Chapter XIII.

Here state the general nature of the problem. Calculation No.

Here state full details of the actual problem.

(A) By Formula $Ay = M\left(\frac{r}{R^N - 1}\right)$ Rule 1.

			Values.	Logs.
Log R^N-1	Log. Ratio (1)	R		
	Multiply Log. R by	N		
	(3)	R^N		
	Convert Log. to ordinary number	R^N		
	deduct unity	-1		
	Log. of this is (4)	$R^N - 1$		
	Log. Amount of Loan	M		
	add Log. r (2)	r		
		$M\ r$		
	deduct Log. $(R^N - 1)$ above (4)	$R^N - 1$		
		Ay		

Required annual instalment, £

(B) By Table III. $Ay = \dfrac{M}{\frac{R^N - 1}{r}}$ Rule 2.

Log. Amount of Loan	M	
Table III. years per cent. Amount of £1 per annum (5)	$\frac{R^N - 1}{r}$	
deduct Log.		
	Ay	

Required annual instalment, £

(C) By Thoman's Table. $Ay = M\left(\frac{a^n}{R^N}\right)$ Rule 3.

per cent. years

Log Amount of Loan	M	
add Log. a^n (8)	a^n	
	$M\ a^n$	
deduct Log. R^N in Table + 10 (3)	R^N	
	Ay	

Required annual instalment, £

The Sinking Fund Instalment.

Table III. Standard Form, 3x.

To find the number of years:

based on Calculation (XV) 1.

Given factors:

Amount of loan ...	M	26495
Annual instalment	Ay	680·234
Rate per cent. ...		$3\frac{1}{2}$
Ratio	R	1·035
Interest of £1 ...	r	0·035

Details of Method:

find	Log. M.	26495	4·4231639
find, and deduct ...	Log. Ay	680·234	2·8326581
difference ...			1·5905058
find, and add... ...	Log. r		$\bar{2}$·5440680
the sum is ...	Log. $R^N - 1$		$\bar{0}$·1345728
find value of this log.		1·36324	
add unity ...		1·	
which is the value of ...	R^N	2·36324	
find	Log. R^N		0·3735087
find	Log. R		0·0149403

To find the number of years, divide the above log. of R^N by the above log. of R, as described in Chapter XXXII, and the quotient is the number of years required, in this case, 25 years.

The Sinking Fund Instalment.

Table III. Standard Form, 3x.

To find the rate per cent:

based on Calculation (XV) 1.

Given factors:

Amount of loan ...	M	26495
Annual instalment	Ay	680·234
Number of years ...	N	25

Details of method;

find	Log. M.	26495	4·4231639
find, and deduct ...	Log. Ay	680·234	2·8326581
difference ...			1·5905058
find value of this log.		38·94986	

which is the amount of loan which will be provided by an annual instalment of one pound for 25 years at the required rate per cent.

To ascertain the rate per cent., refer to Table III, giving the amounts of one pound per annum, and find the nearest value to the above amount of 38·94986 in 25 years. If the rate so found is not near enough, refer to Thoman's tables and find the nearest log. to 1·5905058 which is ascertained by deducting the log. of a^n from the log. of R^N, plus 10.

∴ Required rate per annum, $3\frac{1}{2}$ per cent.

Note. In cases where the rate per cent. is not included in the published tables of compound interest, or in Thoman's tables, the above method will give only approximate results.

Standard Calculation Form, No. 4.

Table IV. To find the present value of an annuity.

Chapter VII.

This form has been used in the solution of problems of the following nature:—

Calculation.

To find the sum now payable which is the equivalent of the future annual sinking fund instalments to be set aside to repay a given loan at the end of a prescribed period of years; and for which such annual instalments might be redeemed ... ... See Chapter XXXII.

Standard Calculation Form, No. 4.

TABLE IV. To find the present value of an annuity.

The following rules are explained at the head of Chapter VII.

Here state the general nature of the problem. Calculation No.

Here state full details of the actual problem.

(A) By Formula. $P = Ay\left(\frac{R^N-1}{R^N\ r}\right)$ Rule 1.

		Values.	Logs.
Log R^N-1	Log. Ratio (1)	R	
	Multiply Log. R by	N	
	(3)	R^N	
	Convert Log. to ordinary number	R^N	
	deduct unity	-1	
	Log. of this is (4)	R^N-1	
	Log. Annuity	Ay	
	add Log. (R^N-1) above (4)	R^N-1	
	deduct Log. R^N above (3)	R^N	
	deduct Log. r (2)	r	
		P	

Required present value, £

(B) By Table IV. $P = Ay\left(\frac{R^N-1}{R^N\ r}\right)$ Rule 2.

Table IV. years per cent. Present Value £1 per annum (6)	$\frac{R^N-1}{R^N\ r}$	
add Log. Annuity	Ay	
	P	

Required present value, £

(C) By Thoman's Table. $P = \frac{Ay}{a^n}$ Rule 3.

per cent. years

Log. Annuity	Ay	
add 10		
deduct Log. a^n (8)	a^n	
	P	

Required present value, £

The Present Value of One Pound per Annum.

Table IV. Standard Form, 4.

To find the number of years:

based on Calculation (XVIII) 14.

Given factors:

Annuity	Ay	40·215
Present value thereof	P	313·118
Rate per cent. ...		3
Ratio	R	1·03
Interest of £1 ...	r	0·03

Details of method:

find	Log. P	313·118	2·4957086
find, and deduct ...	Log. Ay	40·215	1·6043881
difference ...			0·8913205
find value of this log.		7·7861	

which is the present value of an annuity of one pound, at 3 per cent., for the required number of years.

To ascertain the number of years, refer to Table IV giving the present values of one pound per annum, under 3 per cent., and find the nearest value to the above amount of 7·7861. If the rate per cent. is not given in the tables, refer to Thoman's tables, under the nearest rate per cent., and find the nearest log. to 0·8913205, which is found by deducting the log. of a^n there given from 10.

Required period, 9 years.

Note. In cases where the rate per cent. is not included in the published tables of compound interest, or in Thoman's tables, the above method will give only approximate results.

The Present Value of One Pound per Annum.

Table IV. Standard Form, 4.

To find the rate per cent:

based on Calculation in Statement XXXII. D.

Given factors:

Annuity	A	1725·58	
Present value thereof	P	28374·73	
Number of years ...	N	23	

Details of method:

find	Log. P	28374·73	4·4529318
find, and deduct ...	Log. Ay	1725·58	3·2369352
difference ...			1·2159966
find value of this log.		16·4436	

which is the present value of an annuity of one pound for 23 years, at the required rate per cent.

To ascertain the rate per cent. refer to Table IV, giving the present values of one pound per annum, and find the nearest value to the above present value of 16·4436 in 23 years. If the rate so found is not near enough, refer to Thoman's tables and find the nearest log. to 1·2159966, which is ascertained by deducting the log. of a^n there given from 10.

Required rate per annum, 3 per cent.

Note. In cases where the rate per cent. is not included in the published tables of compound interest, or in Thoman's tables, the above method will give only approximate results.

Standard Calculation Form, No. 5.

TABLE V. To find the annuity which a present sum will purchase, or the annuity of which £1 is the present value. Chapter VIII.

This form has been used in the solution of problems of the following nature:—

	Calculation.
To find the equal annual instalment of principal and interest combined, to be paid to the lender, in order to repay a stated loan in a prescribed period	(XII) 4.
To find the amount by which the original annual sinking fund instalment may be reduced in consequence of:—	
(1) a surplus in the fund owing to an excessive past accumulation of the fund	(XVIII) 10.
(2) a surplus in the fund, due to the payment into the fund of any sum not provided out of revenue or rate, namely:—	
(*a*) the proceeds of sale of part of the assets representing the security for the loan	(XVII) 1.
or	
(*b*) a realised profit upon the sale of an investment representing the fund.	
To find the additional sinking fund instalment, to be set aside, and added to the fund during the unexpired portion of the repayment period, to compensate for a deficiency in the amount now in the fund	(XV) 3.

TABLE V. To find the annuity which a present sum will purchase, or of which it is the present value. To find the equal annual instalment of principal and interest combined.

The following rules are explained at the head of Chapter VIII.

Here state the general nature of the problem. Calculation No.

Here state full details of the actual problem.

(A) By Formula. $Ay = P\left(\frac{R^N r}{R^N - 1}\right)$ Rule 1.

			Values.	*Logs.*
Log $R^N - 1$	Log. Ratio	(1)	R	
	Multiply Log. R by		N	
		(3)	R^N	
	Convert Log. to ordinary number		R^N	
	deduct unity		-1	
		(4)	$R^N - 1$	
	Log. Present Sum		P	
	add Log. R^N above	(3)	R^N	
	Log. r	(2)	r	
	deduct Log. $(R^N - 1)$ above	(4)	$R^N - 1$	
			Ay	
	Required annuity, £			

(B) By Table V. $Ay = P\left(\frac{R^N r}{R^N - 1}\right)$ Rule 2.

		Values.	*Logs.*
Table V. years per cent. Annuity £1 will purchase	(7)	$\frac{R^N r}{R^N - 1}$	
add Log. Present Sum		P	
		Ay	
Required annuity, £			

(C) By Thoman's Table. $Ay = P\,a^n$ Rule 3.
per cent. years

		Values.	*Logs.*
Log. Present Sum		P	
add Log. a^n	(8)	a^n	
deduct 10		Ay	
Required annuity, £			

The Annuity which One Pound will Purchase and The Equal Annual Instalment of Principal and Interest Combined.

Table V. Standard Form, 5.

To find the number of years:

based upon Calculation (XV) 3.

Given factors:

Present sum ...	...	P	469·74
Annuity ...	...	Ay	45·594
Rate per cent.	...		$3\frac{1}{2}$
Ratio	...	R	1·035
Interest of £1	...	r	0·035

Details of method:

find	Log. Ay	45·594	1·6589086
find, and deduct ...	Log. P	469·74	2·6718612
difference ...			$\bar{2}$·9870474
find value of this log.		0·097061	

which is the annuity which one pound will purchase at $3\frac{1}{2}$ per cent. for the required number of years.

To ascertain the number of years refer to Table V, giving the annuity which one pound will purchase, under $3\frac{1}{2}$ per cent., and find the nearest value to the above amount of 0·097061. If the rate per cent. is not given in the tables, refer to Thoman's tables, under the nearest rate per cent., and find the nearest log. of a^n to $\bar{2}$·9870474. This may be ascertained by an inspection of the mantissa only.

Required period, 13 years.

Note. In cases where the rate per cent. is not included in the published tables of compound interest, or in Thoman's tables, the above method will give only approximate results.

The Annuity which One Pound will Purchase and The Equal Annual Instalment of Principal and Interest Combined.

Table V. Standard Form, 5.

To find the rate per cent:

based upon Calculation (XVIII) 10.

Given factors:

Present sum	P	447·27
Annuity	A*y*	57·4446
Number of years ...	N	9

Details of method:

find	Log. A*y*	57·4446	1·7592493
find, and deduct ...	Log. P	447·27	2·6505698
difference ...			$\bar{1}$·1086795
find value of this log.		0·128434	

which is the annuity which one pound will purchase for 9 years at the required rate per cent.

To ascertain the rate per cent., refer to Table V, giving the annuity which one pound will purchase, and find the nearest value to the above annuity of 0·128434 in 9 years. If the rate so found is not near enough, refer to Thoman's tables and find the nearest log. of a^n to $\bar{1}$·1086795. This may be ascertained by an inspection of the mantissa only.

Required rate per annum, 3 per cent.

Note. In cases where the rate per cent. is not included in the published tables of compound interest, or in Thoman's tables, the above method will give only approximate results.

Section II.

The Methods of Repayment of the Loan Debt of Local Authorities and Commercial and Financial Undertakings

CHAPTER XI.

THE REPAYMENT OF THE LOAN DEBT OF LOCAL AUTHORITIES AND COMMERCIAL AND FINANCIAL UNDERTAKINGS.

ALTERNATIVE METHODS ALLOWED BY THE PUBLIC HEALTH ACT, 1875, AND OTHER ACTS. COMPARISON OF METHODS AS REGARDS THE ACTUAL REPAYMENT TO THE LENDER, AND THE ANNUAL CHARGE AGAINST REVENUE OR RATE.

The Instalment Method.

BY AN EQUAL ANNUAL INSTALMENT OF PRINCIPAL TO BE REPAID TO THE LENDER. NO SINKING FUND REQUIRED, BUT AN EQUAL PERIODICAL REPAYMENT OF PRINCIPAL. ANNUAL CHARGE AGAINST THE REVENUE OR RATE ACCOUNT OF SUCCESSIVE YEARS COMPOSED OF AN EQUAL AMOUNT OF PRINCIPAL AND A GRADUALLY DECREASING AMOUNT OF INTEREST. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN.

Having obtained a series of rules and formulæ relating to all problems involving compound interest, they will now be applied to problems of actual finance, beginning with the repayment of the loan debt of local authorities. This affords a very good subject for treatment by the mathematical method, not only on account of the variety of the problems occurring in actual practice, but because the original conditions and regulations are of a fairly uniform character. This uniformity is much more pronounced than is the case with the loan debt of commercial and financial undertakings, where not only much more variable and elastic conditions exist, but also greater facilities to alter the original arrangements between the borrower and the lender.

Excluding for the present the provisions contained in early Acts of Parliament, which vary considerably, the general principles now in force, and governing the matter, are contained in Section 234 of the Public Health Act of 1875. These provisions may be accepted as the standard now adopted in all

public general Acts, special Acts, and provisional orders of the Local Government Board. There are of course small variations in detail, but the principle remains in all cases substantially the same, and, with the exception of the introduction of a new form of "non-accumulating" sinking fund, there has not been any change since 1875. These provisions relate to the repayment of the debt, and will apply equally to commercial and financial undertakings, as they are based upon general financial practice. There is not anything new in the methods laid down in the Act, but the merit of the section lies in the fact that for the first time definite methods were prescribed in place of the very varied practice previously followed. To state an extreme case, borrowing powers will never be granted in future without any obligation whatever as to redemption. This section specifies three alternative methods, at the option of the local authority, by which the loan debt may be repaid, and provides in effect that:—

(*a*) The local authority shall repay the moneys so borrowed by:—

(1) equal annual instalments of principal, or by

(2) equal annual instalments of principal and interest combined;

or (*b*) The local authority shall in every year set apart as a sinking fund and accumulate in the way of compound interest such a sum as will, with accumulations in the way of compound interest, be sufficient to pay off the moneys so borrowed within the period sanctioned.

The alternative methods of repayment are usually described as:—

(*a*) (1) The instalment method.

(2) The annuity method.

(*b*) The sinking fund method.

The sinking fund method in the Act of 1875 is the same as the accumulating sinking fund referred to in the 1893 clauses of the Local Government Board.

The sections of the Public Health Act, 1875, and other Acts relating to the borrowing of money by local authorities lay down five distinct principles, namely:—

The power to borrow is limited to works of a more or less permanent nature.

The amount to be borrowed is limited.

The period of repayment must be fixed by the Local Government Board, having regard to the relative permanency of the works.

The Public General Acts contain a provision that the period sanctioned by the Local Government Board for the repayment of any loan shall not in any case exceed a period prescribed in each Act.

The amount annually required to discharge the liability in respect of interest upon the loan and the repayment of the debt is chargeable against the rate or revenue account of each year.

There are two main distinctions to be drawn between the above methods (1) as regards the actual repayment of the loan, and (2) as regards the charge upon the rate or revenue accounts of the successive years of the repayment period. The instalment and annuity methods both provide for the actual repayment to the lender each year of a definite proportion of the loan or of the loan and interest combined. The sinking fund method, on the other hand, contemplates the provision annually of an instalment of such amount as will, if set aside, invested, and accumulated for the prescribed period, provide for the repayment of the loan in one sum at the end of the period. Power is given, however, under certain conditions to apply part of the sinking fund in repayment of the loan during the prescribed period of repayment.

Each of these methods will be considered in detail, taking first:—

The Instalment Method, in which the loan is repaid to the lender by equal annual instalments of principal only, and interest is paid to him upon the balance of loan unpaid. This method applies mainly to advances made to local authorities by the Public Works Loan Commissioners, and also to loans by the larger insurance companies to the Metropolitan boroughs and other local authorities.

This method is also commonly used by commercial and financial undertakings, and is known as the deferred payment system. The hire purchase system, on the other hand, is a commercial form of the annuity method. The instalment method is exceedingly simple in operation, seeing that it is merely an arithmetical calculation, and does not involve any question of compound interest whatever.

Generally the repayment period is 30 years or longer, but in order to simplify the problem and to enable a comparison to be made with the annuity and sinking fund methods to be hereafter described, in all cases the example will relate to the repayment of a loan of £1,000, in ten years, with interest at 5 per cent., which rate will be assumed to be payable to the lender, and will also be the rate of accumulation of the sinking fund.

As will be seen by the following statement, the municipality will repay to the lender at the end of the first year:—

$^1/_{10}$ of the principal	£100	
Interest on £1,000	50	
	——	£150

and this amount will be charged to the rate or revenue account.

At the end of the second year the municipality will repay to the lender:—

$^1/_{10}$ of the principal, as before ...	£100	
Interest on £900	45	
	——	£145

and so on each year until, at the end of the tenth year, they will repay:—

$^1/_{10}$ of the principal, as before ...	£100	
Interest on £100	5	
	——	£105

the effect being, as regards the municipality, that the rate or revenue account will be charged year by year with a gradually decreasing amount, to the relief of the later generations of ratepayers.

As regards the lender, he originally advanced to the municipality a sum of £1,000, which is repaid to him at the rate of £100 per annum, which will require to be invested each year, with the result that his income will be constantly fluctuating, and he will hold a number of small investments instead of one large one. The following statement (XI. A.), shows the operation of the repayment from year to year by means of the constant instalment of £100 of principal, and also shows the decreasing amount of interest annually paid to the lender. As the principal and the interest are both charged to the revenue or rate account, it shows the annually decreasing loan charge.

Similar statements will be given relating to the annuity and sinking fund methods; and, finally, a statement will be prepared

comparing the effect of the repayment by all three methods. In the case of the instalment and annuity methods there is not any accumulating sinking fund, and therefore there is not any complication arising from the rate of accumulation. This will be dealt with under the head of the sinking fund method. The instalment method, so far as regards the actual repayment to the lender, is exactly similar to the ordinary repayment of debt by commercial and financial undertakings, and there is not any difference in principle if the instalments are not equal in amount or are made at unequal intervals of time. The lender receives interest each year, upon the actual balance owing to him, since the last date to which interest has been paid.

Unlike the annuity and instalment methods, there is not any variation in the calculation if the interest be paid half-yearly or otherwise instead of yearly. It is a simple arithmetical calculation not complicated in any degree by compound interest.

STATEMENT XI. A.

The Repayment of Debt of Local Authorities. The Instalment Method.

Showing the repayment of a Loan of £1,000 in 10 years, with interest at 5 per cent. by equal annual instalments of principal only.

Year end.	Owing at beginning of year.	Interest at 5%	Total owing.	Repayments. Principal.	Repayments. Interest.	Repayments. Total.	Balance owing at end of year.	Charge to Revenue. Principal.	Charge to Revenue. Interest.	Charge to Revenue. Total.	Year.
1	1000	50	1050	100	50	150	900	100	50	150	1
2	900	45	945	100	45	145	800	100	45	145	2
3	800	40	840	100	40	140	700	100	40	140	3
4	700	35	735	100	35	135	600	100	35	135	4
5	600	30	630	100	30	130	500	100	30	130	5
6	500	25	525	100	25	125	400	100	25	125	6
7	400	20	420	100	20	120	300	100	20	120	7
8	300	15	315	100	15	115	200	100	15	115	8
9	200	10	210	100	10	110	100	100	10	110	9
10	100	5	105	100	5	105	—	100	5	105	10
	1000	275	—	1000	275	1275	—	1000	275	1275	

CHAPTER XII.

THE REPAYMENT OF THE LOAN DEBT OF LOCAL AUTHORITIES AND COMMERCIAL AND FINANCIAL UNDERTAKINGS (*Continued*).

The Annuity Method,

BY AN EQUAL ANNUAL INSTALMENT OF PRINCIPAL AND INTEREST COMBINED, TO BE REPAID TO THE LENDER.

METHODS OF CALCULATING THE ANNUAL INSTALMENT BY FORMULA AND TABLES; AND THE GENERAL RULES BASED THEREON. NO SINKING FUND REQUIRED, BUT AN EQUAL PERIODICAL REPAYMENT TO THE LENDER OF PRINCIPAL AND INTEREST COMBINED. THE RELATION BETWEEN SUCH EQUAL ANNUAL INSTALMENT, THE SINKING FUND INSTALMENT (CHAPTER XIII) AND THE EQUAL ANNUAL INSTALMENT OF PRINCIPAL ONLY (CHAPTER XI). STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN.

AUTHOR'S STANDARD CALCULATION FORM, NO. 5.

Formulæ.

The whole of the formulæ at the head of Chapter VIII, relating to Table V (the annuity which £1 will purchase) apply to the present method.

General Rules deduced from the formulæ relating to Table V:—

To find the equal annual instalment of principal and interest combined, to repay a given loan during a stated period.

Author's Standard Calculation Form, No. 5.

Rule 1. If the rate per cent. be not given in Table V, or in Thoman's Tables:—

Proceed by the formula relating to Table V.

Calculation (XII) 4 A.

Rule 2. If the rate per cent. be given in Table V:—

Multiply the annuity given in the table by the amount of the loan. The product is the required annual instalment. *Calculation (XII) 4 B.*

Rule 3. If the rate per cent. be given in Thoman's Table:—
To the log. of the amount of the loan add the log. of a^n, as given by Thoman: deduct 10 from the sum of the logs. The remainder is the log. of the required annual instalment. Calculation (XII) 4 C.

Rule 4. Find the sinking fund instalment by any of the rules given in the following chapter; add to the instalment so found one year's interest upon the loan. The rate per cent. in both cases to be the rate of interest to be paid to the lender. The sum is the required annual instalment. Calculation (XII) 5.

To find the rate per cent. or number of years, proceed as shown in the standard form for the purpose, relating to Table V, given in Chapter X.

The Annuity Method. Under this method there is, as in the case of the instalment method, an actual repayment each year to the lender, the whole of which is charged to the rate or revenue account of each year. But in this case the lender receives an equal amount each year, composed of principal and interest combined. To the extent that it involves an equal annual charge upon the rate or revenue account of the municipality during the whole of the repayment period, it is an improvement upon the instalment method, but it still has not any advantage to the lender. As will be seen by the detailed statement, XII. A., following, and also by the comparative statement in Chapter XIII, after the sinking fund method, the annual amount repaid to the lender consists of an increasing amount of principal and a decreasing amount of interest; and, further, if the lender be a trustee, or requires for any purpose to allocate the repayment as between capital and income, he must make a somewhat difficult calculation. The lender has to re-invest each year a gradually increasing amount of principal, unless he sets aside an equal annual proportion of the amount paid to him, as a sinking fund, as will be explained later.

The formulæ and tables will now be applied to ascertain the equal annual instalment of principal and interest combined required to repay a loan of £1,000 in 10 years at 5 per cent. There are several ways of doing this, but the clearest, although not the shortest, will be first described, being the one which best illustrates the principles involved. Leaving the actual calculation for the moment, the transaction will be divided into

two parts, ignoring for the present the annual repayments to the lender. The loan of £1,000, if not repaid, will accumulate at 5 per cent. compound interest with yearly breaks, and at the end of the period will amount to £1628·9, as shown by Calculation (XII) 1. The next step is to ascertain the amount, at the end of 10 years at 5 per cent., of an equal annual instalment of £1 per anuum, or 12·5779, as shown by Calculation (XII) 2.

It has now been ascertained that £1,000 in 10 years at 5 per cent. will amount to £1628·9 and that each £1 of equal annual instalment, or annuity, will at the end of 10 years amount to £12·5779; and it is therefore obvious that the equal annual instalment required will be, in sterling or other currency, exactly the number of times that £12·5779 is contained in £1628·9. By dividing £1628·9 by £12·5779, the required equal annual instalment of principal and interest combined is obtained, viz., £129·51, as shown by Calculation (XII) 3. The actual details of the above Calculations (XII) 1 and (XII) 2 are given at the end of the chapter upon the author's standard calculation forms, No. 1 and No. 3, both of which are made by three methods :—

A. by formula.
B. by the published tables;
and C. by Thoman's tables.

The two factors referred to have now been ascertained :— Calculation (XII) 1 shows that the original loan of £1,000 will in 10 years, at 5 per cent., amount to £1628·90; and Calculation (XII) 2 shows that £1 per annum will in 10 years, at 5 per cent., amount to £12·5779; and the required annual instalment of principal and interest combined is obtained by dividing £1628·90 by £12·5779 by logarithms as follows.

CALCULATION (XII) 3.

To find the equal annual instalment of principal and interest combined to repay a given loan.

Required the equal annual instalment of principal and interest combined, to be repaid to the lender as and when set aside, to repay £1,000 in 10 years at 5 per cent.

By Tables I and III and Logs. Based on Calculations (XII) 1 and (XII) 2.

Table I, Calculation (XII) 1:		
Amount of £1,000 in 10 years at 5 per cent.	1628·9	3·2118930
deduct, log.		
Table III, Calculation (XII) 2:		
Amount of £1 per annum for 10 years at 5 per cent....	12·5779	1·0996079
		2·1122851

which is the log. of the required equal annual instalment of principal and interest combined, viz., £129·51

The principle involved in the above method of ascertaining the equal annual instalment of principal and interest combined, to be repaid to the lender as shown in Calculation (XII) 3, is, that, on the one hand, there is the original loan of £1,000 quietly rolling up all by itself for the prescribed period; and, on the other hand, there is an equal annual instalment of £129·51 also rolling up at the same rate per cent. for the same period. The annual instalment is of such amount that, at the end of the period both accounts will amount to exactly the same sum. Seeing that the rate of accumulation is the same in both cases, it is obvious that the transfer from one account to the other of an annual sum in repayment of principal and interest combined, out of the accumulating credit, may be made without in any way affecting the result arrived at by considering the two factors as independent transactions.

The following table shows the methods of finding the equal annual instalment of principal and interest combined in the foregoing example, and also demonstrates the derivation of the formula relating to Table V from the formulæ relating to Tables I and III:—

	By Published Tables.	Actual Values.	By Formulæ.
Numerator:—			
The amount of £1 in any number of years	Table I	1628·90	R^N
Denominator:—			
The amount of £1 per annum, in the same number of years	Table III	12·5779	$\frac{R^N - 1}{r}$
Table V =	$\frac{\text{Table I}}{\text{Table III}}$	129·51	$\frac{R^N\ r}{R^N - 1}$

In Chapter IX it was pointed out that the above formula $\frac{R^N r}{R^N - 1}$ is the equivalent of Thoman's factor (a^n), both of which denote the annuity which £1 will purchase for any number of years. In Chapter VIII the same formula was arrived at, by deduction, from Table IV, which gives the present values of an annuity of £1. The present example proves that the formula for Table V may be also found by deduction from Tables I and III, and consequently that the equal annual instalment of principal and interest combined may be found by Table V or by Thoman's factor (a^n) in a much more direct way than by using Tables I and III as above. The calculation will therefore be made by Table V, on the author's standard calculation form No. 5, using the three methods therein contained, namely, by formula, by the published table, and by Thoman's method. See Calculation (XII) 4 at the end of this chapter.

The general rules relating to each method are given at the head of this chapter.

The Relation between the Equal Annual Instalment of Principal and Interest Combined, and the Sinking Fund Instalment. On comparing the above Calculation (XII) 3, relating to the annuity method, with Calculation (XIII) 1, in the following chapter by which the sinking fund instalment is ascertained, it will be seen that Calculation (XIII) 1 is the simpler because it involves only one reference to the published tables (No. III.). The present comparison is made in order to compare the annual instalment of principal and interest combined with the sinking fund instalment, ignoring the fact that there is a more direct method of finding the equal annual instalment of principal and interest combined by means of Table V. It will be seen that the equal annual instalment of principal and interest is greater than the sinking fund instalment by £50, which is one year's interest upon the loan of £1,000 at 5 per cent. per annum.

The equal annual instalment of principal and interest combined, to be paid to the lender under the annuity method may be therefore ascertained in the following manner:—

First ascertain the sinking fund instalment which will provide the loan at the end of the period, as in Calculation (XIII) 1, taking as the rate of accumulation of the sinking fund the rate of interest to be paid to the lender under the annuity method.

which is the interest upon £1 for one year. One instalment may also be expressed in terms of the other. If it be required to find the equal annual instalment of principal and interest, having found the sinking fund instalment, divide the formula relating to the equal annual instalment of principal and interest by the formula relating to the sinking fund instalment as follows:—

$$\left(\frac{R^N r}{R^N - 1}\right) \div \left(\frac{r}{R^N - 1}\right) = \left(\frac{R^N r}{R^N - 1}\right) \times \left(\frac{R^N - 1}{r}\right) = \left(\frac{R^N r}{r}\right) = R^N$$

thereby proving that the equal annual instalment, of principal and interest combined, may be found by multiplying the ascertained sinking fund instalment by the value of R^N as given in the published table (No. 1); or by logs., by adding to the log. of the sinking fund instalment the log. of R^N, as given in Thoman's tables. The sum of the above logs. is the log. of the equal annual instalment of principal and interest combined.

Applying this rule to the present example:—

Log. Sinking fund instalment	= 79·5046	1·900 3921
add Log. R^N, 10 years, 5 per cent.	1·62889	0·211 8930
Log. equal annual instalment of principal and interest combined		= 2·112 2851

which, as shown by Calculation (XII) 4, is £129·51.

This method will rarely be used in practice, but is interesting as furnishing a further example of the relation between the above formulæ.

The Repayment of the Loan by the Annuity Method. The following Statement XII. A. shows the repayment of the loan, year by year; and should be compared with the similar Statement XI. A., relating to the instalment method in the previous chapter. It should also be compared with Statement XIII. A., relating to the sinking fund method in the next chapter, when it will be noticed that not only is the total annual charge to the revenue or rate account uniform during the whole of the repayment period under both the annuity and sinking fund methods, but that the total annual charge is also the same in amount in each case provided that the rate of accumulation of the sinking fund is the same as the rate of interest payable upon the loan. The following Statement XII. A. also shows the increasing amounts of principal and

the consequent decreasing amounts of interest contained in the equal annual instalment repaid to the lender.

After considering the sinking fund method in the following chapter the results under the three methods will be shown in tabular form, both as regards the actual repayment to the lender, and also the annual charges to the revenue or rate account during the successive years of the repayment period, in Statement XIII. B.

The two methods already discussed, namely, the instalment method and the annuity method, involve the provision each year, out of rate or revenue of part of the principal and interest, and such annual provision is actually repaid to the lender as and when set aside. They do not therefore in any way partake of the nature of a sinking fund, which relates only to the provision for the repayment of the principal in one amount at the end of a definite period, and will be described in the following chapter.

STATEMENT XII, A.

The Repayment of the Debt of Local Authorities.

The Annuity Method.

Showing the repayment of a Loan of £1,000 in 10 years, with compound interest at 5 per cent. per annum by an equal annual instalment of principal and interest combined.

Year end	Owing at beginning of year	Interest at 5%	Total owing	Repayments			Balance owing at end of year	Charge to Revenue			Year end
				Principal	Interest	Total		Principal	Interest	Total	
1	1000·	50·	1050·	79·51	50·	129·51	920·49	79·51	50·	129·51	1
2	920·49	46·02	966·51	83·49	46·02	129·51	837·	83·49	46·02	129·51	2
3	837·	41·85	878·85	87·66	41·85	129·51	749·34	87·66	41·85	129·51	3
4	749·34	37·46	786·80	92·05	37·46	129·51	657·29	92·05	37·46	129·51	4
5	657·29	32·86	690·15	96·65	32·86	129·51	560·64	96·65	32·86	129·51	5
6	560·64	28·03	588·67	101·48	28·03	129·51	459·16	101·48	28·03	129·51	6
7	459·16	22·95	482·11	106·56	22·95	129·51	352·61	106·56	22·95	129·51	7
8	352·61	17·63	370·24	111·88	17·63	129·51	240·73	111·88	17·63	129·51	8
9	240·73	12·03	252·76	117·48	12·03	129·51	123·25	117·48	12·03	129·51	9
10	123·25	6·27	129·52	123·24	6·27	129·51	nil	123·24	6·27	129·51	10
	1000·	295·10	—	1000·	295·10	1295·10	—	1000·	295·10	1295·10	

Note. The methods by which the lender may apportion the above instalment as between capital and income are described later when dealing with the statement showing the operation of a sinking fund.

Calculation (XII) 1.

Standard Calculation Form, No. 1.

To find the future amount of a present sum.

To find the amount which will be owing at the end of a stated period in respect of a given loan if it be allowed to accumulate at compound interest. Table I.

Required the amount of £1,000 at the end of 10 years at 5 per cent. per annum, compound interest.

(A) By Formula. $A = P R^N$ Rule 1, Chapter IV.

Log R^N { Log. Ratio	R	1·05	0·0211893
Multiply Log. R by	N	10	10
	R^N	$(1·05)^{10}$	0·2118930
Log. Present Sum	P	1000	3·
add Log. R^N above	R^N		0·2118930
	A		3·2118930

Required future amount, £1628·90.

(B) By Table I. $A = P R^N$ Rule 2, Chapter IV.

Table I. 10 years, 5 per cent.			
Amount of £1	R^N	1·628895	0·2118930
add Log. Present Sum	P	1000	3·
	A		3·2118930

Required future amount, £1628·90.

(C) By Thoman's Table. $A = P R^N$ Rule 3, Chapter IV.
5 per cent. 10 years.

Log. Present Sum	P	1000	3·
add Log. R^N	R^N		0·2118930
	A		3·2118930

Required future amount, £1628·90.

Calculation (XII) 2.

Standard Calculation Form, No. 3.

To find the amount of an annuity in any number of years.

Table III.

Required the amount of £1 per annum for 10 years at 5 per cent. per annum, compound interest.

(A) By Formula. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 1, Chapter VI.

Log $R^N - 1$: Log. Ratio	R	1·05	0·0211893
Multiply Log. R by	N	10	10
Convert Log.	R^N	$(1·05)^{10}$	0·2118930
to ordinary number	R^N	1·6289	
deduct unity	−1	1·	
Log. of this is	$R^N - 1$	0·6289	$\bar{1}$·7985779
Log. Annuity	Ay	1·	0·
add Log. $R^N - 1$ above	$R^N - 1$		$\bar{1}$·7985779
	$Ay(R^N - 1)$		$\bar{1}$·7985779
deduct Log. r	r		$\bar{2}$·6989700
	M		1·0996079

Required amount, £12·5779.

(B) By Table III. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 2, Chapter VI.

Table III. 10 years, 5 per cent.	$R^N - 1$	12·5779
Amount of £1 per annum	r	
Add Log. Annuity	Ay	
	M	

Required amount, £12·5779. This amount is given in Table III.

(C) By Thoman's Table. $M = Ay\left(\frac{R^N}{a^n}\right)$ Rule 3, Chapter VI.

5 per cent. 10 years.

Log. Annuity	Ay	1·	0·
add Log. R^N in Table + 10	R^N		10·2118930
	$Ay\ R^N$		10·2118930
deduct Log. a^n	a^n		9·1122851
	M		1·0996079

Required amount, £12·5779.

Calculation (XII) 4.

Standard Calculation Form, No. 5.

To find the annuity which a present sum will purchase for any number of years.

To find the equal annual instalment of principal and interest combined to repay a given loan. The Annuity Method.

Table V.

Required the equal annual instalment of principal and interest combined to be repaid the lender as and when set aside, to repay £1,000 with interest in 10 years at 5 per cent.

(A) By Formula. $Ay = P\left(\frac{R^N r}{R^N - 1}\right)$ Rule 1, Chapter VIII.

Log $R^N - 1$	Log. Ratio	R	1·05	0·0211893
	Multiply Log. R by	N	10	10
		R^N	$(1·05)^{10}$	0·2118930
	Convert Log. to ordinary number	R^N	1·62889	
	deduct unity	-1	1·	
		$R^N - 1$	0·62889	$\bar{1}$·7985779
	Log. Present Sum	P	1000	3·
	add Log. R^N above	R^N		0·2118930
	Log. r	r	0·05	$\bar{2}$·6989700
				1·9108630
	deduct Log. $(R^N - 1)$ above	$R^N - 1$		$\bar{1}$·7985779
		Ay		2·1122851

Required annuity, £129·5046.

(B) By Table V. $Ay = P\left(\frac{R^N r}{R^N - 1}\right)$ Rule 2, Chapter VIII.

Table V. 10 years, 5 per cent. Annuity £1 will purchase	$\frac{R^N r}{R^N - 1}$	0·1295	$\bar{1}$·1122851
add Log. Present Sum	P	1000	3·
	Ay		2·1122851

Required annuity, £129·5046.

(C) By Thoman's Table. $Ay = P\, a^n$ Rule 3, Chapter VIII.
5 per cent. 10 years.

Log. Present Sum	P	1000	3·
add Log. a^n	a^n		9·1122851
			12·1122851
deduct 10	Ay		2·1122851

Required annuity, £129·5046.

CHAPTER XIII.

THE REPAYMENT OF THE LOAN DEBT OF LOCAL AUTHORITIES AND COMMERCIAL AND FINANCIAL UNDERTAKINGS (*Continued*).

The Sinking Fund Method.

BY SETTING ASIDE AND ACCUMULATING AN EQUAL ANNUAL INSTALMENT IN ORDER TO PROVIDE THE PRINCIPAL ONLY AT THE END OF THE REDEMPTION PERIOD.

1. The Accumulating Sinking Fund.

METHODS OF CALCULATING THE ANNUAL INSTALMENT BY FORMULA AND TABLES AND THE GENERAL RULES BASED THEREON. DESCRIPTION OF THE METHOD AND THE CALCULATION OF THE ANNUAL INSTALMENT. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN. COMPARISON OF THE INSTALMENT, ANNUITY AND SINKING FUND METHODS, ILLUSTRATED BY A STATEMENT SHOWING IN EACH CASE THE ANNUAL CHARGE TO REVENUE OR RATE.

AUTHOR'S STANDARD CALCULATION FORM, No. 3x.

2. The Non-accumulating Sinking Fund.

THE OBJECT OF THE FUND AND ITS RELATION TO THE METHODS PRESCRIBED IN SEC. 234 OF THE PUBLIC HEALTH ACT, 1875. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN AND THE ANNUAL CHARGES TO REVENUE OR RATE.

NOTE.—UNLESS IT IS OTHERWISE EXPRESSLY STATED, THE TERM "SINKING FUND," WILL, THROUGHOUT THE BOOK, APPLY ONLY TO AN ACCUMULATING SINKING FUND.

Formulæ.

VARIATION OF TABLE III. *The annuity which will amount to £1 in any number of years, or* $\dfrac{1}{\textit{Table III.}}$

A. To find the annuity which will amount to £1 in any number of years:—

(1) *Formula,* $$Ay=\left(\frac{r}{R^N-1}\right)$$

by logs.: *Log. (required annuity)=Log.* r − *Log.* (R^N-1)

(2) *By Thoman's Method:—*

Formula, $$Ay=\frac{a^n}{R^N}$$

by logs.: *Log. (required annuity)=Log.* a^n − *(Log.* $R^N+10)$

B. To find the annual sinking fund instalment which will amount to any given loan, in any number of years:—

(1) *Formula,* $$Ay=M\left(\frac{r}{R^N-1}\right)$$

by logs.: *Log. (required instalment)=Log. of Loan+Log.* r − *Log.* (R^N-1)

(2) *By. Thoman's Method:—*

Formula, $$Ay=M\left(\frac{a^n}{R^N}\right)$$

by logs.: *Log. (required instalment)=Log. of Loan+Log.* a^n − *(Log.* $R^N+10)$

General Rules deduced from the above formulæ.

To find the annual instalment to be set aside and accumulated as a sinking fund to repay a given loan at the end of a prescribed number of years. Author's Standard Calculation Form, No. 3x.

Rule 1. If the rate per cent. be not given in Table III, or in Thoman's Tables:—

Proceed by the formula derived from Table III, as shown above. Calculation (XIII) 1 A.

Rule 2. If the rate per cent. be given in Table III:—

Divide the amount of the loan by the amount given in the table. The quotient is the required annual instalment. Calculation (XIII) 1 B.

*Rule 3. If the rate per cent. be given in Thoman's Table:—
To the log. of the loan, add the log. of a^n as given by Thoman. Deduct therefrom the log. of R^N as given by Thoman; also deduct 10. The remainder is the log. of the required instalment.*

Calculation (XIII) 1 C.

To find the rate per cent. or number of years, proceed as shown in the standard form for the purpose, given in Chapter X.

The Accumulating Sinking Fund. The sinking fund method provides for the setting aside each year, and accumulating by way of compound interest, such a sum as will be sufficient to pay off the money borrowed within the prescribed period. It will be gathered from the above provision (which is laid down in the Public Health Act, 1875, and is contained in principle in all subsequent Acts) that this method differs from the instalment and annuity methods in two particulars, viz.:—

1. It provides for the repayment of principal only, and is quite apart from any question of interest on the loan.
2. The repayment of the principal money is not made by instalments, but takes place at the end of the prescribed period, with certain reservations which will be dealt with later.

In both the instalment and annuity methods there is not any question of the rate of accumulation, as the annual repayments are made direct to the lender, and there is not therefore any sinking fund set aside.

In the case of the annuity method as applied to the repayment of the debt of a local authority, the lender may, or may not, be able to reinvest the increasing proportion of principal included in the annual instalment paid to him, at the calculated rate which he receives upon his investment, but this does not enter into the calculation in any way. So far as the local authority is concerned they undertake to pay to the lender interest at the agreed rate for such period only during which they have the use of the money. As regards sinking funds relating to the loan debt of commercial and financial undertakings, this is also generally the case, but the purchaser of an annuity may require that the annual instalment shall be fixed at such an amount as will yield him a specified rate of interest upon his principal, and at the same time enable him to reinvest the annual repayments of principal at a lower rate than he receives as interest, in order to replace the capital.

With regard to the provision in the Public Health Act that the annual sum set apart shall be sufficient, *after paying all expenses,* to pay off the money borrowed within the period sanctioned, it is found in practice that the expenses, being of uncertain amount, cannot be calculated actuarially. They are therefore omitted from the calculation, and if small in amount are charged direct to the rate or revenue account as and when incurred. Where the expenses of raising the loan are large in amount, as is the case when the loan is authorised by special Act of Parliament, the Act generally provides that the cost of obtaining the powers shall be repaid by means of a separate sinking fund to mature in a short period, generally 5 to 10 years.

The sinking fund method is the one now generally adopted by all local authorities for the annual provision for redemption of debt. It is called a sinking fund when it relates to loans, a loans fund when it relates to the annual provision of principal and the payment of dividends on stock, and a redemption fund when it relates to stock issued under the stock regulations of the Local Government Board. This is all very misleading and confusing, but these are the statutory terms. The general term sinking fund, with some distinguishing word added, would better describe the nature of the fund which fulfils the same purpose both in the case of loans and stock. The sinking fund relates only to the ultimate repayment of principal by means of an equal annual sum charged against the year's revenue or rate, such annual sum being set aside and accumulated by investment in outside securities. With regard to the interest payable upon the loan, it is obvious, since no provision is made for it in the sinking fund instalment, that during the whole of the period of repayment the rate or revenue account of each year will be charged with interest upon the full amount of the original loan, and this notwithstanding the fact that part of the sinking fund may have been applied in the redemption of part of the loan before the expiration of the repayment period.

Since the interest paid upon the loan is quite outside the question of the sinking fund, the rate of accumulation of the fund may, and generally does, differ from the rate of interest payable to the lender. Section 234 (5) of the Public Health Act, 1875, provides that the local authority may apply the whole or any part of the sinking fund in the repayment of the debt, but if they do so they must pay into the sinking fund annually a sum equivalent to the interest which would have been produced by that part of the sinking fund so applied. This provision, which is generally inserted in all general and

special Acts, is absolutely necessary. The sinking fund is calculated to accumulate at a definite rate per cent., and if any part of the fund be used to repay part of the debt the fund will be deficient to that amount, and will lose the interest upon the portion of the fund so applied. This provision is equal to saying that any such application of the sinking fund shall be treated as an investment of the fund as if it had been actually invested in outside securities.

The section provides that the local authority shall pay into the sinking fund a sum equivalent to the interest which would have been produced by that part of the fund applied towards the redemption of debt. But in practice it is usual to estimate that the sinking fund will accumulate at a lower rate per cent. than the interest paid upon the loan. This is in order to provide for a fall in the rate of interest obtainable upon first-class investments, and it results in a larger annual instalment being set aside than would be the case if the sinking fund were calculated to accumulate at the higher rate of interest paid upon the loan. The general practice, when loans are redeemed out of the sinking fund, is to pay into the fund the actual amount of interest previously paid to the loan holders. Any surplus thus arising helps to make up the deficiency caused by the low rate of interest obtained when part of the sinking fund is in the bank awaiting investment, as often happens.

With regard to the investment of the sinking fund until it is applied in the redemption of debt, it was until recently the practice of Parliament and also of the Local Government Board to require that it should be invested in outside securities, but of late years Parliament has given power under special Acts to invest the sinking funds in the stocks and loans of the same local authority. The sinking fund, however, cannot be invested in any other department of the same authority unless that department has obtained statutory powers to borrow the amount, and is therefore under a statutory obligation to set aside out of revenue or rate a sinking fund for its redemption.

In the case of local authorities issuing stock at par which afterwards commands a premium, the whole of the cost of any part of the stock which is redeemed at a premium cannot be taken out of the sinking fund, but only the par value of the stock, the premium being charged to the rate or revenue account at the time the stock is redeemed. If such purchases at a premium are variable, both as to time and amount, they may be dealt with by means of a supplementary sinking fund relating to the premium only, in such a manner that the premium is

spread equally over the unexpired period. If the premium is fixed at the date of issue of the stock it should be included in the original sinking fund calculation, but if the stock at any time commands a premium beyond this amount the method of providing for it in advance will be more difficult.

THE CALCULATION OF THE ANNUAL INSTALMENT. The actual calculation will now be considered. The instalment is required to be set aside annually and accumulated at compound interest in order to provide the principal sum only, and the question of interest upon the loan does not enter into the calculation. Under these conditions it would appear that the calculation is much simpler than in the annuity method, using Tables I and III, although not so if Table V be used. The question to be solved, therefore, is, taking as before a loan of £1,000 repayable at the end of 10 years at 5 per cent., "what annuity accumulated at 5 per cent. for 10 years will at the end of that period amount to £1,000"? This rate per cent. is the rate of accumulation of the sinking fund and not the rate of interest payable upon the loan. All questions involving the calculation of the amount of an annuity are treated by the formula relating to Table III, already referred to, namely,

$$M = Ay\left(\frac{R^N - 1}{r}\right)$$

the actual values for £1 per annum being given in Table III.

The sinking fund calculation may be compared with that made in the case of the annuity method, Calculation (XII) 3, in which the instalment was required to provide a sum equal to the "amount" of £1,000 accumulated at 5 per cent. compound interest.

In this case the instalment has to provide only the capital sum of £1,000 without interest. Consequently if the actual loan be taken instead of the "amount" of the same sum at the end of the period, as in Calculation (XII) 3, the required annual instalment will be obtained for the reasons given in discussing Calculation (XII) 1. The rule, therefore, to find the sinking fund instalment is:—

> "*Divide the amount of the loan by the amount of £1 per annum as given in Table III for the required number of years at the stated rate per cent. and the quotient is the required annual instalment.*"

The problem resolves itself into the following:—

If £1 per annum in 10 years at 5 per cent. will, at the end of that period, amount to £12·5779, what annuity will, under the same conditions, amount to £1,000?

The required formula is obtained by transposing the formula relating to Table III as follows:—

$$Ay = \left(\frac{\frac{M}{R^N - 1}}{r}\right) \text{ or } Ay = M\left(\frac{r}{R^N - 1}\right)$$

and the calculation will be made upon the author's standard form, No. 3x, by the three methods previously referred to.

It may be interesting to point out that this calculation is an example of how the use of a formula may lead to the discovery of another method of making the same calculation. It will be noticed in the above case that the numerator in the formula is $M \times r$, (which means that £1,000 has been multiplied by 0·05) and the result divided by $(R^N - 1)$. But £1,000 × 0·05 = £50, which is the interest upon £1,000 for one year at 5 per cent., and therefore that an alternative rule may be stated as follows:

"*To ascertain the sinking fund instalment, find the interest upon the amount of the loan for one year at the sinking fund rate of accumulation (not the rate of interest payable upon the loan) and divide by* $(R^N - 1)$, *which is the actual value given in Table I, reduced by unity.*"

This rule is not of any practical advantage over those given at the head of this chapter, and will not therefore be further considered.

THE FINAL REPAYMENT OF THE DEBT BY THE OPERATION OF THE SINKING FUND. The following statement shows the final repayment of the loan by the operation of the sinking fund and also the annual payment of interest upon the whole of the loan until the end of the prescribed period when the accumulation of the fund is equal to the amount of the loan which is then repaid the fund exhausted, and the annual contributions cease.

This statement shows that the fund is increased annually by the instalment provided out of revenue or rate and by the income received upon the investment of previous instalments. This income from investments is the amount which the lender, under the annuity method, would have obtained if he had taken out of each annual instalment of £129·51 paid to him the sum of £50 by way of interest upon his loan, and invested the

remaining £79·51 and the subsequent accumulations at 5 per cent. annually to provide his capital at the end of the term. He would by this means obtain a more regular income than by treating as income the interest shown in the tables relating to the annuity method, which decreases year by year. It will further be noticed that the interest charged to the revenue or rate account under the annuity method, as shown in the table relating to that method, added to the income received from investments, as shown in the table relating to the sinking fund method, are together equal in each year to £50, which is the interest paid to the lender annually under the sinking fund method. See Statement XIII. A., page 139.

If, therefore, the lender, under the annuity method, requires to equalise his annual income, he may do so by setting aside an equal annual sum out of the instalment and accumulating it as a sinking fund to provide his capital.

This mode of equalising the income might be adopted by trustees and executors with the object of securing a fixed income for a tenant for life, but will apply only to an annuity for a fixed term. The above argument is, however, subject to the reservation that the lender may not be able, year by year, to reinvest the periodical repayments of principal to yield the rate per cent. upon which the annual instalment was based.

Comparison of the Three Methods. It is now possible to compare the repayment of loans by instalment, annuity and sinking fund methods, as above described, and this will be done from the standpoints both of the lender and borrower by means of the following statement (XIII. B., page 140.

In Chapter XI, the instalment method has been compared with the annuity method, and it is interesting to compare the annuity method with the sinking fund method. In each case the annual instalment is ascertained by dividing a definite sum by the same accumulated amount of an annuity of £1 for 10 years at 5 per cent., but in the case of the annuity method the amount so divided is the amount of the principal sum accumulated at compound interest, whilst in the sinking fund method the amount so divided is the principal sum itself without accumulations. This is owing to the fact that the annual instalment in the case of the annuity method includes interest, whereas the annual instalment in the case of the sinking fund relates to the principal sum only. The annual instalment in the sinking fund method, therefore, is smaller than in the case of the annuity method.

By the annuity method, Calculation (XII) 3, the instalment of principal and interest is	£129·51
By the sinking fund method the instalment of principal only is	£79·51
The difference being one year's interest on £1000 at 5 per cent.	£50·00

Under the sinking fund method, therefore, the total annual charge to revenue or rate in respect of principal and interest is exactly equal year by year to the total annual charge under the annuity method, viz. £129·51 in each case.

This has already been referred to in discussing the annuity method in Chapter XII.

With regard to the instalment method the total annual charge to revenue or rate account in respect of principal and interest is greater in the earlier years and is gradually reduced from £150 to £105 in ten years. The relative merits of the annuity method and the sinking fund method as regards the annual incidence of local taxation are equal and are more equitable than the instalment method. As regards the investor, under the instalment method he receives a decreasing annual payment made up of a constant amount of principal and a decreasing amount of interest; but he has definite knowledge of how much is interest and how much is principal. Under the annuity method he receives an equal annual payment made up of an increasing amount of principal and a decreasing amount of interest; but without an elaborate calculation he is unable to apportion the amount paid to him between capital and income. Under both the instalment and the annuity methods the investor receives annual sums in respect of his capital which he has to reinvest in small amounts.

Comparing the sinking fund method, on the one hand, with the instalment and annuity methods on the other, from the point of view of the investor, it will be seen that under the sinking fund method he receives each year an equal amount by way of interest upon his money, and has the further advantage of a permanent investment of the whole of his capital for a definite long term. If he wishes to realise he has a definite security to place upon the market either to be bought by some other investor or to be redeemed by the local authority out of the sinking fund. Under the sinking fund method he has to run the risk of a fall in the market value in the case of a loan

raised by the issue of stock; but, on the other hand, he may realise a profit. Summing up the respective merits of the various methods of repayment of the debt of local authorities, it may fairly be concluded that the accumulating sinking fund method is by far the best. It bears equally upon the taxation or revenue of each year of the repayment period; and as regards the investor, it is at once more convenient and more equitable than either of the other two methods.

The Non-Accumulating Sinking Fund. Up to this point the enquiry has been limited to accumulating sinking funds similar to the one prescribed in the Public Health Act, 1875. The principal feature of such a fund is the provision out of revenue or rate of an equal annual instalment to be set aside and accumulated for a prescribed period at a rate per cent. to be fixed in anticipation, with as near approach to accuracy as can be obtained. In the case of loans with long repayment periods this is very difficult, and it therefore becomes necessary to compare the actual amount in the fund periodically with the calculated amount which should be in the fund as shown by the pro forma account. Any surplus or deficiency in an accumulating fund should be credited to, or charged against, the revenue or rate account of each year, but this entails considerable labour, and it is one of the objects of the non-accumulating sinking fund to avoid this by providing an automatic accurate accumulation of the fund irrespective of the rate of income received on the investments representing the fund. The basis of the method is the instalment system discussed in Chapter XI. where each year a definite sum is charged to the revenue or rate account and repaid to the lender. The annual instalment in the case of the non-accumulating sinking fund is calculated precisely as in the instalment method, namely, by dividing the amount of the loan by the number of years in the repayment period. But in this case the annual instalment of principal is not repaid to the lender, but is invested by the local authority in order to provide the amount of the loan at the end of the period. Since an equal amount is added to the fund year by year it requires merely an arithmetical calculation to ascertain the amount which should be in the fund at any time. Seeing that the total amount of the loan is provided by the actual equal annual charges to revenue or rate, it is obvious that the income arising from the investments representing the fund need not be added to the fund. On comparing the actual instalments only,

under the instalment method in Chapter XI, with those under the accumulating sinking fund method in this chapter, it will be seen that the charge to revenue or rate under the instalment method is greater than under the accumulating sinking fund method, consequently in the case of a non-accumulating fund the income to arise from the investments may be credited to the rate or revenue account to which the annual instalment of principal has been debited. The excess of the original annual instalment in the non-accumulating fund over the instalment in the accumulating fund will not be compensated by the reduction therein due to the income received from the investment of the fund, because such resulting income will be small during the earlier years; and an equality in the annual burden will not be reached until the end of the fifth year out of ten. The first four years will therefore bear an additional burden, and the last five years will be relieved, as compared with the annual incidence under the accumulating fund, in a similar manner to the instalment method. As regards the ratepayer, the non-accumulating fund will have all the disadvantages of the instalment method already pointed out. The lender, on the contrary, will be in a better position, since he obtains a permanent investment and is relieved of the periodical reinvestment of small amounts of capital. The actual method by which the local authority provides the sinking fund has not any particular interest to him.

As between the instalment and annuity methods, on the one hand, and the two sinking fund methods on the other, the only difference is the date at which he shall be repaid, and he invests in the particular loan which best meets his requirements. Under the two periodical repayment methods the lender may be said to keep his own sinking fund, whereas in both sinking fund methods the local authority does this for him. There are not any mathematical principles involved in the non-accumulating fund, but it is merely an arithmetical one. The following table shows the final repayment of the loan by the operation of the fund. In order that it may be compared with the accumulating sinking fund it has been assumed that the investments yield 5 per cent. per annum, and that no part of the fund is applied in redemption of debt during the period:—

STATEMENT XIII. C.

The Repayment of the Debt of Local Authorities.

The Non-Accumulating Sinking Fund.

Showing the repayment of a loan of £1,000, at the end of 10 years by an equal annual instalment of principal, to be set aside and invested as a sinking fund, the annual income upon the investments being applied in reduction of subsequent instalments. Interest at 5 per cent.

Year end	Annual Instal-ment	Deduct Income received	Net charge to revenue or rate	Interest on loan	Total charge to revenue or rate	Amount in fund	Year end
1	100	—	100	50	150	100	1
2	100	5	95	50	145	200	2
3	100	10	90	50	140	300	3
4	100	15	85	50	135	400	4
5	100	20	80	50	130	500	5
6	100	25	75	50	125	600	6
7	100	30	70	50	120	700	7
8	100	35	65	50	115	800	8
9	100	40	60	50	110	900	9
10	100	45	55	50	105	1000	10

The clauses authorising the non-accumulating sinking fund contain the usual permission to apply part of the fund in redemption of debt; but in this case there is not any necessity to have regard to the interest which would have been received in respect of the part of the fund so applied, because, although the income received from the investments will be smaller in consequence of such application, yet the interest payable upon the loan will be correspondingly reduced, and there will not therefore be any alteration in the combined charge for interest and redemption. There is another matter which may properly be considered to the advantage of the non-accumulating fund, if the greater burden imposed during the earlier years is not fatal to its adoption. This affects the possible and very probable variation in the rate of income to be received from the investments representing the fund. In the case of the accumulating fund, as already pointed out, it very rarely happens that the

fund increases in accordance with the calculated amount, thus rendering it necessary to make frequent adjustments through the revenue or rate account. As will be shown in later chapters, the variation in the rate per cent., whether of income from investments or of accumulation, gives rise to many of the problems which have to be dealt with. The non-accumulating fund has this advantage, that any variation in the rate per cent. is at once automatically adjusted, seeing that if from any cause there is a fall in the rate of income from investments there is a corresponding increase in the charge to revenue or rate due to the decreased relief to subsequent annual instalments afforded by the amount of income received from the investments.

In the case of a non-accumulating fund the annual instalment of the full amount would be credited each year to the sinking fund account and not debited direct to the revenue or rate account, but to an intermediate account which might be called the "non-accumulating sinking fund suspense account." This suspense account would be credited with the income received from the investments representing the fund, and with interest allowed by the bank, if any, whether any part of the fund had been applied in redemption of debt or not, and it would be debited with the interest actually paid or accrued on the outstanding loan. The balance remaining to the debit of this suspense account would then be charged to the revenue or rate account and would represent the total charge against the year in respect of loan indebtedness. The accounts of an accumulating sinking fund might be kept in a similar manner. A "sinking fund interest suspense account" would be opened, and the sinking fund account would be credited and the suspense interest account debited with the actual amount of interest which should yearly accrue to the fund as shown by the pro forma account. The suspense interest account would be credited with the actual income received from the investments, and bank interest, and the balance, either debit or credit, but generally debit, would be closed by transfer to the revenue or rate account. By this means the sinking fund accounts would always stand at their proper calculated amounts, any reduction in income would immediately become apparent and there would be no possibility of the gradual accumulation of many small deficiencies requiring at some future time considerable correction by an increased annual instalment.

STATEMENT XIII, A.

THE REPAYMENT OF THE DEBT OF LOCAL AUTHORITIES.

THE SINKING FUND METHOD. THE ACCUMULATING FUND.

Showing the Repayment of a Loan of £1,000 at the end of 10 years by an Annual Instalment to be set aside and accumulated as a Sinking Fund at 5 per cent. per annum.

	LOAN.						SINKING FUND.				
Year end	Owing at beginning of year	Interest at 5%	Total owing	Interest paid	Principal repaid	Balance owing at end of year	Balance at beginning of year	Interest received thereon	Annual Instalment	Total	Year end
1	1000	50	1050	50	nil	1000	—	—	79·51	79·51	1
2	1000	50	1050	50	,,	1000	79·51	3·97	79·51	162·99	2
3	1000	50	1050	50	,,	1000	162·99	8·15	79·51	250·65	3
4	1000	50	1050	50	,,	1000	250·65	12·53	79·51	342·69	4
5	1000	50	1050	50	,,	1000	342·69	17·13	79·51	439·33	5
6	1000	50	1050	50	,,	1000	439·33	21·96	79·51	540·80	6
7	1000	50	1050	50	,,	1000	540·80	27·04	79·51	647·35	7
8	1000	50	1050	50	,,	1000	647·35	32·37	79·51	759·23	8
9	1000	50	1050	50	,,	1000	759·23	37·96	79·51	876·70	9
10	1000	50	1050	50	1000	nil	876·70	43·79	79·51	1000·00	10
	—	—	—	—	—	—	—	204·90	795·10	—	

The total annual charge to revenue or rate is the instalment	£79·51
and the annual interest paid on the loan	50·00
	£129·51

STATEMENT XIII, B.

The Repayment of the Debt of Local Authorities.

General Summary. All Methods.

Showing the Annual charge to Revenue or Rate in respect of a Loan of £1,000 repayable in 10 years under the Instalment, Annuity, and Sinking Fund methods.

	Instalment Method.			Annuity Method.			Sinking Fund Method.			
Year.	Principal.	Interest.	Total.	Principal.	Interest.	Total.	Principal.	Interest.	Total,	Year.
1	100	50	150	79·51	50·	129·51	79·51	50	129·51	1
2	100	45	145	83·49	46·02	129·51	79·51	50	129·51	2
3	100	40	140	87·66	41·85	129·51	79·51	50	129·51	3
4	100	35	135	92·05	37·46	129·51	79·51	50	129·51	4
5	100	30	130	96·65	32·86	129·51	79·51	50	129·51	5
6	100	25	125	101·48	28·03	129·51	79·51	50	129·51	6
7	100	20	120	106·56	22·95	129·51	79·51	50	129·51	7
8	100	15	115	111·88	17·63	129·51	79·51	50	129·51	8
9	100	10	110	117·48	12·03	129·51	79·51	50	129·51	9
10	100	5	105	123·24	6·27	129·52	79·51	50	129·51	10
	1000	275	1275	1000·	295·10	1295·11	795·10	500	—	

Note. In the case of the non-accumulating sinking fund, Statement XIII, C, the total annual charges to revenue or rate will be the same year by year as under the instalment method, provided that the local authority is able to invest the equal annual sum set aside, to yield a rate of interest equal to the rate payable upon the loan.

Calculation (XIII) 1.

Standard Calculation Form, No. 3x.

To find the annual sinking fund instalment. Table III.

Required the annual instalment to be set aside and accumulated as a sinking fund at 5 per cent. to provide £1,000 at the end of 10 years.

(A) By Formula. $Ay = M\left(\frac{r}{R^N - 1}\right)$ Rule 1, Chapter XIII.

Log $R^N - 1$	Log. Ratio	R	1·05	0·0211893
	Multiply Log. R by	N	10	10
		R^N	$(1·05)^{10}$	0·2118930
	Convert Log. to ordinary number	R^N	1·6289	
	deduct unity	-1	1·	
	Log of this is	$R^N - 1$	0·6289	$\bar{1}$·7985779
	Log. Amount of Loan	M	1000	3·
	add Log. r	r	·05	$\bar{2}$·6989700
		M r		1·6989700
	deduct Log. $(R^N - 1)$ above	$R^N - 1$		$\bar{1}$·7985779
		Ay		1·9003921

Required annual instalment, £79·5046.

(B) By Table III. $Ay = \dfrac{M}{\frac{R^N - 1}{r}}$ Rule 2, Chapter XIII.

Log. Amount of Loan	M	1000	3·
Table III. 10 years, 5 per cent. Amount of £1 per annum *deduct* Log.	$\frac{R^N - 1}{r}$	12·5779	1·0996079
	Ay		1·9003921

Required annual instalment, £79·5046.

(C) By Thoman's Table. $Ay = M\left(\frac{a_n}{R^N}\right)$ Rule 3, Chapter XIII.

5 per cent., 10 years.

Log. Amount of Loan	M	1000	3·
add Log. a^n	a^n		9·1122851
	M a^n		12·1122851
deduct Log. R^N in Table + 10	R^N		10·2118930
	Ay		1·9003921

Required annual instalment, £79·5046.

Section III.

Sinking Fund Problems

The Annual Instalment

CHAPTER XIV.

SINKING FUND PROBLEMS.

RELATING TO :—

(1) THE AMOUNT IN THE FUND.

(2) THE RATE PER CENT :—

(a) OF INCOME TO BE RECEIVED UPON THE PRESENT INVESTMENTS REPRESENTING THE FUND ;

(b) THE FUTURE RATE OF ACCUMULATION.

(3) THE REDEMPTION PERIOD.

(4) THE RATE PER CENT. AND THE REDEMPTION PERIOD IN COMBINATION.

Definition of terms :

1. THE PRESENT INVESTMENTS.
2. THE ANNUAL INCREMENT.

THE PRO FORMA ACCOUNT. Having discussed the several alternative methods of repayment of loan debt by local authorities laid down by statute, and having described the methods of finding the annual sums to be set aside for that purpose out of revenue or rate, the subject will now be considered in its practical aspect. Most of these transactions extend over very long periods and all trace of the original calculation is often lost. It is therefore advisable in all cases involving a periodical provision for repayment by means of a sinking fund to prepare at the outset a pro forma account showing how the calculated annual instalment should work out during the whole of the period.

The Local Government Board auditors in many cases require this to be done in respect of all loans coming under their supervision, and it is a practice to be commended and followed. Such a pro forma account enables a comparison to be made annually between the actual and the calculated working out of the fund so that any discrepancy may be immediately set right. Especially does this apply to a deficiency caused by part of the sinking fund lying uninvested in the bank and earning less than

the calculated rate per cent. of accumulation or due to a general decrease in such rate. Any such deficiency will be of small amount in any one year, and may be charged against the revenue or rate account of the particular year, so keeping the sinking fund up to the proper amount. But cases have arisen in which this has not been done, and from the above, and other causes, the amounts in the sinking funds have been seriously deficient. In such cases it becomes necessary to ascertain the proper amount which would have been in the fund if the original anticipations had been realised. This is a contingency which may arise in the case of a local authority, and there are other questions with regard to sinking funds which, although not affecting local authorities, yet are very important in connection with the sinking funds of commercial and financial undertakings.

NATURE OF PROBLEMS. In dealing with all cases of adjustment of a sinking fund it will be necessary to refer continually to the present state of the fund as the basis upon which all such adjustments are made, and later, when dealing with other problems, it will be seen that the present position of the fund plays an equally important part. Such questions will be considered later, and will comprise:—

(1) A deficiency in the fund. Chapters XV, XVI.

(2) A surplus in the fund. Chapters XVII, XVIII.

(3) A variation in the rate per cent. at which the fund was originally expected to accumulate. Chapter XIX, etc.

(4) A variation in the rate of income to be yielded by the investments representing the fund. Chapters XX, XXVII.

(5) A variation in the repayment period. Chapter XXIV.

(6) A variation in the repayment period accompanied by a variation in the rate of accumulation. Chapter XXVI.

Any or all of the above contingencies may have to be taken into account in an adjustment, and as they arise only after the fund has been in operation for part of the original repayment period, it is important to ascertain exactly the position of the fund at the time the adjustment is required to be made. It is generally the case with the sinking funds of local authorities that the amount standing to the credit of the fund is required

to be invested in specific outside securities allocated to the fund, or, which is the same in effect, shall have been applied in part repayment of the original loan. In the case of commercial and financial undertakings it is usual to impose the obligation of such outside investment in order to ensure that the original purpose of the fund shall be carried out, and that the amount in the fund shall be actually available for the repayment of the debt at the end of the period. Any enquiry therefore into the adequacy or otherwise of the amount in the fund at any time will properly include, not only the value of the investments representing the fund at the present time, but also an enquiry as to the probable value at the end of the repayment period. It will be necessary to ascertain whether they are yielding or are likely to continue to yield a return by way of income equal to or differing from the calculated rate per cent. of accumulation. In the following chapters, treating of the above possible causes of variation, as far as possible for purposes of convenience and comparison, the position of an imaginary sinking fund will be ascertained at the end of the 12th year of an original period of 25 years, and the position of the fund will be shown at that time, when the enquiry and any subsequent rectification is made, in the following terms, viz.:—

(1) *The value of the present investments* representing the amount in the fund.

(2) *The present annual increment* at the time the enquiry is made, and before the rectification to meet the new conditions.

Present Investments. The term "present investments" will be used to denote the value of the investments representing the amount which actually stands to the credit of the fund and not the amount which should so stand by calculation at the original rate of accumulation as shown by the pro forma account. In fixing the precise market value regard should be had to the probability of the individual investments ultimately yielding the original cost price, and if any fall in value has occurred, or is likely to occur, it should as far as possible be included in the adjustment. In dealing with a surplus or a deficiency in the fund, any actual change in value should be taken into account in calculating the amended annual instalment; but where the problem concerns the period of repayment or the rate of accumulation, and especially if the fund has a long unexpired period to run, it is hardly possible to make

any exact forecast of the future value of the investments, or of the future rate of income to be received therefrom, and this should be provided for by making an allowance when deciding upon the amended rate of accumulation, namely, by taking it at a slightly lower rate than would otherwise be sufficient. In the whole of the following examples, except a deficiency or a surplus in the fund, it will be assumed that the fund stands at the exact amount shown by the original calculation; and, further, that the various investments representing the fund are each worth now the exact amount paid for them, and will be so at the end of the period. This will sufficiently explain without further reference the meaning attached to the term "present investments" in the following pages.

The Annual Increment. With regard to the annual increment, it will be seen, on considering the sinking fund at its inception, that there is then only one factor to deal with, namely, the repayment of a definite loan (or the provision of a definite sum) at the end of a stated number of years. This term will be referred to in the following pages as the "period of repayment or redemption," and in order to make the adjustment it is necessary to fix an average rate per cent. at which the future payments to the fund may reasonably be expected to accumulate by subsequent investment. It is very difficult, if not impossible, to do this correctly in the case of a fund having a long period of repayment, and the practice generally is to assume a rate of accumulation slightly lower than the rate of interest payable to the loanholders. This will allow for a fall in the accumulation rate owing to fluctuations of the money market or for a deficiency in the income of the fund caused by delay in finding an investment which leaves money idle in the bank, earning only a low rate of interest. If the annual deficiency in the income of the fund or any annual surplus be small it should be rectified, as and when it arises, by adjusting it by means of the revenue or rate account, but if the annual deficiency or surplus be large, it is better to adjust the annual instalment immediately in the manner to be described later under the head of variation in the rate of accumulation. Having fixed the future estimated rate of accumulation, the calculation is made in the manner shown in Calculation (XIII) 1, to ascertain the annual instalment to be set aside each year to accumulate at the estimated rate. This annual instalment thus becomes the annual increment during the first year, but after the first instalment has been invested another factor is introduced

into the annual increment, namely, the income from the investments representing the fund.

It is not often that any question affecting the adequacy of the amount in the fund arises during the earlier years of the repayment period. Generally it is much later, and in the following examples it has been taken as the 12th year of a period of 25 years. By this time the fund will have amounted to a large proportion of the total sum to be ultimately provided, and the accruing annual income from investments will (with a $3\frac{1}{2}$ per cent. rate of accumulation) be about one-half of the original annual instalment. Any adjustment of the fund at the end of the 12th year will therefore depend largely upon the future rate of income to be yielded by the present investments representing the fund. And this adjustment may actually be rendered necessary by a fall in the rate of income yielded by the present investments, occurring at a time when the rate yielded by other investments of all kinds is also falling. If the original rate of accumulation be likely to be maintained in spite of a fall in the income received from the present investments, there is not any need, as shown in Chapter XX (variation B, in the rate per cent. of income) to make any adjustment by calculation in the annual instalment. All that is required is to take an additional annual sum out of revenue or rate, equal to the amount of the reduction in the future annual income to be received from the present investments, and the fund will continue to accumulate as originally calculated. But where, as in Chapter XXI (variation C in the rate per cent.) it is necessary at the same time to provide for a reduction in the rate of income from the present investments as well as a reduction in the rate of accumulation, the problem becomes more complicated because there are then two different rates per cent. acting upon two different factors. The rate of income upon the present investments has no relation to the annual instalment provided out of revenue or rate which is acted upon by the accumulation rate only. But the actual amount (if not the rate per cent.) of the income from investments is also acted upon by the accumulation rate, and it is possible to state definitely the annual sum which will be received in respect of such income. Consequently, the difficulty attending the two rates per cent. may be avoided by treating the future income from the present investments as an annuity certain which will continue to be received during the whole of the unexpired portion of the repayment period in exactly the same way as the original annual instalment will continue to be set aside out of

revenue or rate. These two annual factors together will be considered as constituting the future annual increment to be included as an asset in the adjustments, and to be supplemented, as will be seen later, by any additional annual instalment (to be provided out of revenue or rate) which may be found necessary to make up for the decrease in the income from the present investments, and also any further deficiency caused by a reduction in the rate of accumulation. This supplemented annual increment will be referred to later as the future or amended annual increment, as defined in Chapter XXII. Although in the examples which will be considered later a reduction in the rate of income from investments will be assumed, it is quite possible that there may be an increase in the rate of income, which would have the effect of reducing the original annual instalment instead of increasing it. It rarely happens, however, that there is an increase in the rate of accumulation. It is unwise to predict a change which will have the effect of relieving the present revenue or rate account to the possible detriment of future years, and if any surplus in the fund arises in this way it is usually dealt with at the time. The above remarks will explain the reason for the methods adopted later of showing the position of the fund at the end the 12th year when dealing with variations in the rates per cent. of accumulation and income which differ from the methods adopted to show the position at the end of a similar period when dealing with a deficiency or a surplus in the amount in the fund or with variations in the period of repayment without any variation in the rate per cent. either of income or of accumulation. In both the latter cases (see Statements XV. B. and XXIV. A.), which do not involve any variation in the rate of accumulation or in the rate of income, the assets of the fund include the accumulated amount (using the term as in Table I) of the value of the present investments at the end of the respective repayment periods. This amount includes the present value of the investments (£9463 and £9932·74) and the accruing compound interest, because they both accumulate at the same rate which is the same as the rate of income upon the investments.

But in problems involving a variation in the rate per cent. of accumulation, without any variation in the rate of income from investments (as in Statement XIX. A.) it is necessary to find the future amount of the present investments by two calculations because whilst the present investments continue to yield $3\frac{1}{2}$ per cent. per annum, the income so yielded accumu-

lates at only 3 per cent. It is therefore requisite to include the present value of the investments, viz., £9932·74 and to add thereto the sum to which the annual income will accumulate at the end of the period at the amended accumulation rate. As above remarked, it is not necessary to consider the annual increment in connection with problems involving a variation in the rate of income from investments only, but later in Chapter XXVI, when dealing with problems involving a variation in the period of repayment complicated by a variation in the accumulation rate, the annual increment again becomes an important factor. The annual increment has been considered in this exhaustive manner because it is a convenient way of expressing the resulting correction required in consequence of any of the above variations.

It is the adjusted annuity under the amended conditions which is the equivalent of the original annuity under the previous conditions. It may be divided, at both periods, into its component parts of:—

(1) The income from the present investments received from outside sources, and

(2) The annual instalment, to be provided out of revenue or rate, which is the object of enquiry in all cases.

The term will be found very useful when dealing with all actual adjustments, since by dividing the accretions to the fund, as between income from outside investments and contributions from internal revenue, a clearer insight is obtained into the principles underlying the methods adopted.

Methods of Adjustment, based upon the Annual Increment.
(1) *The Annual Increment (ratio) Method*. It will be gathered from the previous remarks that an adjustment in a sinking fund due to any variation in the original conditions may be made in terms of the annual increment, and that there is a definite relation always existing between the annual increment before adjustment (the present annual increment) and the annual increment after the necessary adjustment has been made (the future or amended annual increment). These terms are fully defined at the head of Chapter XXII, where the component parts of each annual increment are exactly described. In both cases the annual instalment may be found by deducting from the annual increment the income from the present investments, thereby eliminating from the calculation any variation in the

rate per cent. of income from investments, and confining the enquiry to the variation in the rate of accumulation only. The annual increment may be considered as a simple annuity to be set aside for a number of years (N) and accumulated at a rate per cent. per annum expressed by the factor (R) or ratio, and the combination of these factors, as regards an annuity or other periodic sum is expressed by the formula $\frac{R^N - 1}{r}$, the derivation of which, from the simple formula $A = P\ R^N$, is fully described in Chapter VI. There is an exact ratio always existing between a given annuity to be accumulated for a stated number of years at a stated rate per cent., and the equivalent annuity to be accumulated for a varying number of years, at a varying rate per cent., depending upon the respective values of N and R.

This is the basis of the annual increment (ratio) method, which is fully described in Chapter XXII, and which has been used in many of the examples in the following chapters.

(2) *The Annual Increment (balance of loan) Method.* In all problems involving an adjustment in a sinking fund there are two fixed factors to be considered, namely:—

(1) The amount of loan to be ultimately repaid, and

(2) The amount now standing to the credit of the fund represented by the present investments.

And in addition there are two variable factors, namely:—

(1) The future period of repayment (N years).

(2) The future rate of accumulation of the fund expressed by the factor (R) or ratio.

Any variation in the future rate of income to be received upon the present investments representing the fund has already been eliminated by merging such annual income in the annual increment.

In all problems involving a variation in the original conditions governing a sinking fund the subject of inquiry is the future amended annual obligation, and this may be ascertained by reducing the present factors to a common basis, namely, the balance of original loan which will be unprovided if the amount now in the fund be immediately applied in redeeming an equivalent part of the loan ultimately repayable. The balance of loan, thereby unprovided for, represents the accumulated amount of an annuity equal to the future or amended annual increment to be set aside for the unexpired or substituted repayment period and accumulated at the original

or varied rate of accumulation. This balance of loan may be ascertained by deducting from the amount of loan ultimately repayable the amount now in the fund as represented by the present investments; and the future annual obligation, which is the future annual increment, may be ascertained by calculating, on standard form, No. 3x, the sinking fund instalment required to provide that amount under the altered conditions, both as regards the period of repayment and the rate per cent. of accumulation. The amended annual increment so ascertained does not, however, represent the amount to be charged annually against the revenue or rate account of the local authority. The conditions governing a sinking fund, as laid down in section 234 (5) of the Public Health Act, 1875, provide that if at any time during the operation of a sinking fund any part of such fund be applied in redemption of debt, the local authority shall, out of its annual rate, pay into the sinking fund a sum at least equal to the amount of interest which would have accrued to the fund if such amount had not been so applied. Consequently the future amended annual instalment is found by deducting from the future or amended annual increment, ascertained in the above manner, the annual income to be received upon the present investments which have been considered as having been immediately applied in the redemption of an equivalent part of the loan, whether the rate of income upon such investments remains unaltered or is varied.

This is the basis of the annual increment (balance of loan) method, which is fully described in Chapter XXII, and which has been used in many of the examples in the following chapters

CHAPTER XV.

SINKING FUND PROBLEMS RELATING TO THE AMOUNT IN THE FUND.

A deficiency in the fund; how it may arise and how it may be adjusted.

PRELIMINARY CALCULATION OF A TYPICAL SINKING FUND TO BE USED TO ILLUSTRATE THE PROBLEMS TO BE DISCUSSED IN THE FOLLOWING CHAPTERS. METHODS OF ASCERTAINING THE POSITION OF A SINKING FUND AT ANY TIME. A DEFICIENCY IN THE FUND AND THE VARIOUS WAYS IN WHICH IT MAY BE CORRECTED. GENERAL SUMMARY OF METHODS OF ADJUSTMENT.

Before considering in detail the various problems arising in connection with a sinking fund it should be stated that there are in each case several methods of making the required adjustment, all of which depend upon the present position of the fund, and the future variation in the original conditions. The subsequent enquiry will include variations in all the fundamental factors relating to such a fund, namely, the amount of the fund, the period of repayment of the loan, the rate of accumulation of the fund, and the future rate of income to be received upon the present investments representing the fund. All these factors have each their own effect upon the ultimate function of the fund, namely, the repayment of the loan, but in addition they act and react one upon the other.

For the purpose of comparison, therefore, each of the possible variations will be considered in relation to one and the same fund, and it will be necessary to treat all the problems on, as far as possible, parallel lines, with the result that in the first instance the most direct method of making the adjustment will not be discussed, although it will be afterwards fully described. The first subject of enquiry will relate to the simple problem of a deficiency in the amount in the fund without any further complication, and the adjustment of such a deficiency will be made by the deductive method, to be followed later when dealing with other matters affecting the fund.

The following is a summary of the general rules as to the adjustment of a deficiency in a sinking fund where the amount in the fund only is in question, and the period of repayment,

the future rate of income upon the present investments, and the future rate of accumulation all remain unaltered. In this chapter a deficiency in the fund has been treated in a very exhaustive manner, perhaps more so than is due to its relative importance. This course has been purposely adopted in order to demonstrate the practical relation between the various formulæ and the tables deduced therefrom.

Summary of the methods of adjustment.

Variation I (Deficiency), in which the adjustment is made by an additional annual instalment to be set aside during the whole of the unexpired portion of the original repayment period.

Method I. THE DEDUCTIVE METHOD, *based upon all the factors governing the fund.* *Statement XV. B.*

(1) *Calculate the amount which should stand to the credit of the fund; being the accumulation, at the calculated rate, of the annual instalments which should have been set aside.* *Calculation (XV) 2. £9932·744.*

(2) *Ascertain the value of the present investments representing the fund, including in the case of a local authority, the loan repaid by means of the sinking fund.* *£9463·00.*

(3) *The difference between the above amounts so found will be the deficiency or surplus in the amount of the fund at the time of making the enquiry.* *£469·744.*

(4) *Calculate the amount to which the value of the present investments (as in 2) will accumulate at the end of the original repayment period.* *Calculation (XV) 4. £14799·71.*

(5) *Calculate the amount of the remaining original annual sinking fund instalments at the end of the same period.* *Calculation (XV) 5. £10960·62.*

(6) *Deduct the sum of the two amounts so obtained (£25760·33) from the amount of the original loan.*

(7) *The difference represents the amount of loan which will be unprovided for in the case of a deficiency, or provided for in excess, in the case of a surplus, at the end of the original repayment period (actually £734·659).* *£734·67.*

(8) *Calculate the additional annual sinking fund instalment required to provide this sum at the end of the repayment period.* *Calculations (XV) 3 and (XVI) 1. £45·594.*

(9) *Adjust the original sinking fund instalment by adding to it the annual instalment so obtained in (8) in the case of a deficiency or by deducting it in the case of a surplus.*

(10) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XVI. A.*

(11) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired portion of the repayment period for future reference.* *Pro forma Account, No. 2, Chapter XVI.*

Method II. *In which the original instalment does not enter into the calculation.* *Statement XVI. A.*

(1) *Calculate the amount to which the sum which should be in the fund, as found by Calculation (XV) 2, will accumulate at the end of the repayment period.*
Calculation (XVII) 2. £15534·375.

(2) *Calculate, and deduct from the sum so found, the amount to which the value of the present investments (£9,463) will accumulate at the end of the repayment period.*
Calculation (XV) 4. £14799·71.

(3) *The difference will be the amount of original loan which will be unprovided for in the case of a deficiency or provided for in excess in the case of a surplus (as found in No. 7, Method 1). (actually £734·659.)* *£734 665.*

(4) *Adjust the original instalment, as in Nos. 8 and 9 in Method 1, above.*

(5) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XVI. A.*

(6) *Prepare a pro forma account, as mentioned above.*
No. 2, Chapter XVI.

Method III. THE DIRECT METHOD, *based entirely upon the present position of the fund.* *Statement XVI. A.*

(1) *Calculate the amount which should stand to the credit of the fund, being the accumulation at the calculated rate, of the annual instalments which should have been set aside.* *Calculation (XV) 2. £9932·74.*

(5) *Deduct therefrom the future annual income to be received from the present investments.* *£331·205.*

(6) *The remainder will be the amended annual instalment to be provided out of revenue or rate instead of the original instalment.* *£725·828.*

The difference between the two instalments will be the additional annual instalment found by either of the preceding methods. *£45·594*

(7) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XVI. A.*

(8) *Prepare a pro forma account, as mentioned above.* *No. 2, Chapter XVI.*

Note. Calculations (XV) 1 and (XV) 2 are given in full at the end of this Chapter. The remainder are given, in an abbreviated form, in the Appendix.

CALCULATION OF A TYPICAL SINKING FUND. The previous chapter contains a brief summary of the nature of the problems likely to arise with regard to sinking funds of all kinds both in connection with local authorities and commercial or financial undertakings. There may be at times a combination of the several variations, but in the first instance each problem will be considered alone, deferring the examination of more complicated cases. In order to do this in a consecutive manner, an imaginary sinking fund will be adopted which will be used to illustrate the whole of the examples to be afterwards considered, because by this means only is it possible to apply the results obtained in considering the simpler problems, to those of a more complex nature. It will be assumed that the sinking fund is in respect of a loan of £26,495, payable at the end of a period of 25 years, and that the instalments will be set aside annually, and will accumulate by investment at $3\frac{1}{2}$ per cent. per annum. The first step is to ascertain the annual instalment, and the calculation will be made upon standard calculation form No. 3x, by the three methods described at the head of Chapter XIII. See Calculation (XV) 1. In this case, as in all others, the method by formula is shown because although the methods by table, including Thoman's, are much shorter, yet all the published tables contain only a limited number of rates per cent.

Further, the tables are not of much assistance when it is necessary to ascertain the rate per cent. or the number of years with accuracy, which can only be done by the method by formula, and then sometimes only approximately. Anyone depending upon the published tables alone without a knowledge of the method by formula is at a great disadvantage when the book of published tables is absent. An acquaintance with the methods by formula and a table of logs. is all that is required, and a very small memorandum book will contain the whole of the formulæ mentioned in this work, which will be found at the head of the various chapters, and also in Chapter X dealing with the standard calculation forms prepared by the author. There is a further advantage gained by a knowledge of the formulæ and how they are arrived at, namely, a clear understanding of the principles underlying the theory of compound interest which renders it an easy matter to make all calculations by one or more alternative methods and thereby prove the accuracy of the results obtained. In making a calculation similar to the foregoing in which it is necessary to multiply or divide a large principal sum by a figure containing 5 places of decimals it is important to be extremely careful to obtain the exact logs. or antilogs. by means of the tables of proportional parts which will be found in the margin of the log. tables. In the above instance, and in all other cases where it is required to find the log of R^N, the log. of R should be carefully ascertained, especially as to the last 3 or 4 figures. In order to obtain the Nth power of R, the log. of R is multiplied by 25, and any error in the last two figures will have a material effect upon the result so found by multiplication. For this reason, in Table V. (A.), in Chapter V, containing the values of (R) for various rates per cent. the corresponding logs. of (R) are given to eight places instead of seven as in the usual log. tables. These logs. may be multiplied by the number of years and the seventh figure adjusted, leaving out the eighth figure. The logs. of R^N are given in Thoman's tables for many rates per cent., and even in cases where the method by formula is used it may be taken direct from Thoman's tables with a saving of time. The logs. of (R^N-1) cannot be found from the tables, but only by calculation, although the actual values of R^N-1 may be found by deducting unity from the actual values given in Table I.

Calculation (XV) 1 shows that an annual instalment of £680·234 is required, and the pro forma account No. 1 at the end of this chapter shows the normal accumulation of the fund.

METHOD OF ASCERTAINING THE POSITION OF A SINKING FUND AT ANY TIME. Having ascertained that an annual sinking fund instalment of £680·234 is required to be set aside and accumulated at $3\frac{1}{2}$ per cent. per annum for 25 years to repay a loan of £26,495, at the end of that period, this information will now be applied to an enquiry into the position of the fund at the end of the 12th year. In an investigation of this nature occurring in actual practice the annual instalment would of course be the basis of the enquiry as it would have been in operation for a period of 12 years. The first stage of the actual enquiry is to ascertain the amount which should now stand to the credit of the fund, on the assumption that the annual instalment has been regularly set aside and has been promptly invested at the end of each year, to yield $3\frac{1}{2}$ per cent. This amount, as shown by Calculation (XV) 2, should be £9932·744. The next step is to ascertain the actual amount standing to the credit of the fund in the books of the local authority or private undertaking and then to compare this amount with the actual value of the investments representing the fund, including in the case of a local authority the loans redeemed by means of the fund. In the case of a commercial or financial undertaking there may not be any obligation to invest the fund in specific outside securities, and the amounts to be charged annually against the profits of the concern may be allowed to remain uninvested and go to swell either the floating or fixed assets. In such a case it may, and will most probably, happen that the book-keeping has been correct, and that the profit and loss account of the undertaking has been each year charged with the proper annual instalment and also with the proper annual interest upon the increasing balances to the credit of the fund. Under such conditions there will rarely be any necessity for enquiry seeing that the fund will always stand in the books at the correct amount, and any deficiency of assets representing the fund will not be apparent, but will be merged in the general state of the assets of the concern. But in the case of commercial and financial undertakings, where there is an obligation to take the amount of the annual instalments out of the floating assets of the concern and invest the same in specific outside securities, the case is exactly similar to the conditions imposed by Parliament upon all local authorities, and may be treated on precisely similar lines. The deficiency in both cases may arise in two ways, even if the annual instalments have been regularly set aside and the proper amount of money actually paid into the sinking fund account. The first cause of the deficiency may be

that owing to delay in investing the instalments, or owing to a fall in the rate of income received from the investments, the fund has not accumulated at the rate originally anticipated and upon which the calculation of the original annual instalment was based. The second cause of the deficiency may be that the investments have depreciated in value and cannot now be considered as representing the amount standing to the credit of the fund, and there may have been in addition an actual loss on realisation. But it is necessary to go further and ascertain whether these investments will, or will not, as far as can be judged, be of such a value at the end of the repayment period that they will fulfil the original purpose of redeeming their proportion of the loan. As already remarked in dealing with the present investments in Chapter XIV, this is a very difficult matter if the unexpired repayment period is a long one; and it is therefore the general practice to assume the future estimated rate of accumulation on the low side, leaving any further adjustment to be made at a later date when the conditions will be better known. In the case of local authorities, as will be seen by a perusal of Article 11 (2) of the County Stock Regulations of 1891, the Local Government Board are empowered to take cognisance of such matters, and the same supervision may be said to apply to the whole of the loans of local authorities. In the case of commercial or financial undertakings the adequacy or otherwise of these investments and of the fund generally would be investigated by the auditors of the company or by or on behalf of the loan holders. In the present chapter it will be assumed that there is a deficiency in the sinking fund of a definite amount arising from any of the above causes, but for the present the problem will not be complicated in any way by a variation in the period of repayment or in the future rates per cent. of income or of accumulation.

The Various Methods of Correcting a Deficiency in a Sinking Fund. Having assumed that there is now an actual ascertained deficiency in the sinking fund the various methods will now be considered by which it may be made good. In the case of a local authority such a deficiency may often arise, but generally it is of small amount due entirely to a reduction in the rate of income on part of the fund uninvested and in the bank. In practice this is met by charging any such deficiency to the general revenue or rate account of each year. If the deficiency in the case of a local authority is large, owing either to serious omissions in previous years or to the accumulation of

many small annual deficiencies, the matter would be decided by the Local Government Board or by Parliament when next powers are sought by special Act. This need not now be discussed in detail because all the available methods will be fully described later. Taking actual figures, it will be assumed that the above imaginary sinking fund (requiring an annual instalment of £680·234 to repay £26,495 in 25 years at an accumulation rate of $3\frac{1}{2}$ per cent.) amounts at the end of

12 years to	£9463·000
instead of the correct amount shown by Calculation XV. (2)	£9932·744
or a deficiency of	£469·744

and that the conditions governing the fund require that this deficiency should be made good in some manner out of rate or revenue, or out of profits in the case of a commercial or financial undertaking.

There are several ways in which such a deficiency may be corrected, namely:—

(*a*) By an immediate payment of the deficiency of £469·744 into the fund, which need not, however, be considered, because, although the soundest financially, it has no bearing upon the subject under review.

(*b*) By an additional annual sinking fund instalment to be spread over the whole of the unexpired 13 years of the original repayment period, in augmentation of the original annual instalment of £680·234.
(Variation I, Chapter XVI.)

(*c*) By an additional annual sinking fund instalment to be spread over a shorter period than the full unexpired term of 13 years. (Variation II, Chapter XVI.)

Having dismissed the correction by an immediate payment into the fund, the last two alternatives will be applied to the imaginary deficiency in order to ascertain the corrected annual instalment consequent thereon. The above deficiency of £469·744 represents an amount of money payable now, being the amount (in the sense in which it is used in Table III) of past annual omissions accumulated at $3\frac{1}{2}$ per cent. It does not represent an equivalent amount of the original loan, as shown later by Calculation (XV) 6. Stated in terms of the original loan, it is the present value at $3\frac{1}{2}$ per cent. per annum of £734·659, part of that loan, repayable in 13 years from the present time, the

repayment of which has not in the past been provided for as it should have been.

The several methods of adjusting the deficiency given in summary form at the head of this chapter will now be described in detail, commencing with the direct method, III, which is the simplest, after which Method I will be considered, followed by Method II, leaving Method IV to be dealt with in the following chapter.

METHOD III. The present deficiency of £469·744, if not complicated by other varying factors of time or rate per cent., may be regarded in its simplest form as the present value of an additional future annual instalment required to be set aside and accumulated during the unexpired portion of the original repayment period in augmentation of the original instalment; and in the summary of methods at the head of this chapter this is described as the direct method No. III. The additional annual instalment is found by Calculation (XV) 3, which shows that the deficiency of £469·744 is the present value of an additional annual instalment of £45·594 to be set aside and accumulated at $3\frac{1}{2}$ per cent. during the unexpired 13 years of the original repayment period. The same result is obtained by Calculation (XVI) 1 in the following chapter, which shows that the annual instalment which will amount to £734·659 of original loan at the end of the period is also £45·594. The above amount (£734·659) of original loan (by Calculation (XV) 6 in this chapter) is shown to be the accumulated amount of the present deficiency of £469·744.

METHOD I. The investigation will now be continued on the lines set out in Method I, at the head of this chapter. The present position of the fund may be stated in terms of the present value of each of the component parts of the fund, namely, the present investments, the deficiency, and the remaining original annual instalments. Seeing, however, that the object of the fund is to repay the loan, and that other causes of adjustment all affect the ultimate amount of the loan, the effect will be more clearly shown by reducing the whole of the factors in all cases to terms of loan, repayable at the end of the prescribed period. This will require three calculations, as follows:—(1) Ascertain the sum to which the present investments (£9,463) will accumulate at the end of the unexpired period of 13 years at $3\frac{1}{2}$ per cent. See Calculation (XV) 4. (2) Add to this amount the sum to which the remaining original

annual instalments of £680·234 will amount at the end of the same period, also accumulated at $3\frac{1}{2}$ per cent. See Calculation (XV) 5. The sum of these two factors will represent the reduced portion only of original loan which would be provided if the present deficiency were not corrected. This total added to the amount of £734·659 to which the present deficiency of £469·744 would accumulate in 13 years at $3\frac{1}{2}$ per cent. [see Calculation (XV) 6] will make up the total amount of the original loan. This last factor is the measure of the deficiency expressed in terms of original loan, and may be treated in the same way as the full amount of the loan, in Calculation (XV) 1, to find the original annual instalment. The required annual instalment so found, namely, £45·594, represents the additional annual sum to be set aside and accumulated in augmentation of the original annual instalment of £680·234. See Calculation (XVI) 1.

The three calculations to show the equivalent amounts of original loan will be made as before by formula and logs., and also by Table III and Thoman's tables. There is really not any necessity to prove the result by further calculation because the above results, added together, should be equal to the total amount of original loan to be provided at the end of the prescribed period.

METHOD II. As already stated in the summary at the head of this chapter, the sum of £734·659, being the amount of loan which will remain unprovided if the present deficiency be not corrected, may also be ascertained by leaving out of account the future original annual instalments (which, *per se*, are unaffected by any present deficiency in the fund), and comparing the ultimate accumulated amount, at the end of the period, of the present investments of £9463·00 with the accumulated amount of the sum of £9932·744 which, as shown by Calculation (XVII) 2, should have been in the fund if the original anticipations had been realised, as follows:—

The ultimate amount of the present investments of £9463·00, as shown by Calculation (XV) 4, will be	£14799·710
and, in Calculation (XVII) 2, it is shown that the above sum of £9932·744 will in 13 years at $3\frac{1}{2}$ per cent. amount to	£15534·375
a difference of (actually £734·659)	£734·665

which, as proved by Calculation (XV) 6, is the ultimate amount of loan represented by the present deficiency of £469·744. The following summary will make the matter clear:—

A Deficiency in the Fund. **Statement XV. A.**

Showing the relation between the calculated amount which should be in the fund, the value of the present investments representing the fund, the deficiency in the fund and the respective equivalent amounts of original loan to be provided at the end of the period of repayment.

	Calculation.	Calculated amount in the fund.	Calculation.	Actual Value of Present investments.	Calculation.	Balance being deficiency
Amounts at the end of the 12th year...	(XV) 2	9932·744	..	9463·00	—	469·744
Amounts of original loan which will be provided by the accumulation of the above for 13 years at $3\frac{1}{2}$ per cent. ...	(XVII) 2	15534·375	(XV) 4	14799·710	(XV) 6	734·659

Calculation (XV) 3 shows that the above deficiency of £469·744 is the equivalent present value of an annual instalment of £45·594, which, accumulated for 13 years at $3\frac{1}{2}$ per cent., will, as shown by Calculation (XVI) 1 in the following chapter, provide the above portion namely £734·659, of the original loan.

The following Statement XV. B. shows the present position of the fund, and also the amount of loan which will be provided at the end of the unexpired portion of the repayment period, namely, 13 years, by the accumulation of the amount now standing to the credit of the fund to be increased by the remaining original annual instalments, but without any correction being made to adjust the present deficiency of £469·744. The final repayment of the loan after correcting the present deficiency by an additional annual instalment will be shown in Statement XVI. A. in the following chapter.

A Deficiency in the Fund. Statement XV. B.

The Deductive Method. No. 1.

Showing the position of the fund at the end of the 12th year, and the amount of loan which will be unprovided at the end of the repayment period if the present deficiency be allowed to accumulate, instead of being immediately corrected by an additional annual instalment.

Present investments (at end of 12th year) £9463·00

Amount thereof, accumulated for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 4 £14799·71

Original annual instalment :—

Amount of £680·234 per annum, for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 5 £10960·62

Provision already made will repay loan of ... ... £25760·33

Deficiency, being the balance of loan unprovided for, represented by the present deficiency of £469·744, accumulated for 13 years at $3\frac{1}{2}$ per cent. (actually £734·659) Calculation (XV) 6 734·67

Amount of original loan ... ... £26495·00

Additional annual instalment required,

Calculations (XV) 3 and (XVI) 1 £45·594

Amended annual instalment,

Original annual instalment	£680·234
Additional annual instalment	45·594
	£725·828

The final repayment of the loan by the operation of the sinking fund after making the above adjustment in the annual instalment is shown in Statement XVI. A., and by the pro forma account, No. 2, Chapter XVI.

Pro forma Sinking Fund Account. No. 1.

Loan of £26,495, repayable at the end of 25 years.

Annual Instalment. Calculation (XV) 1. £680·234.
Rate of Accumulation, $3\frac{1}{2}$ per cent.

Showing the normal accumulation of the fund.

Year.	Amount in the fund at beginning of year.	Income received from investments $3\frac{1}{2}$ per cent.	Annual sinking fund instalment.	Amount in the fund at end of year.	Year.
1	Nil	Nil	680·234	680·234	1
2	680·234	23·808	680·234	1384·276	2
3	1384·276	48·450	680·234	2112·960	3
4	2112·960	73·954	680·234	2867·148	4
5	2867·148	100·350	680·234	3647·732	5
6	3647·732	127·671	680·234	4455·637	6
7	4455·637	155·947	680·234	5291·818	7
8	5291·818	185·213	680·234	6157·265	8
9	6157·265	215·504	680·234	7053·003	9
10	7053·003	246·853	680·234	7980·090	10
11	7980·090	279·302	680·234	8939·626	11
12	8939·626	312·884	680·234	9932·744	12
13	9932·744	347·648	680·234	10960·626	13
14	10960·626	383·622	680·234	12024·482	14
15	12024·482	420·857	680·234	13125·573	15
16	13125·573	459·395	680·234	14265·202	16
17	14265·202	499·282	680·234	15444·720	17
18	15444·720	540·565	680·234	16665·519	18
19	16665·519	583·293	680·234	17929·046	19
20	17929·046	627·517	680·234	19236·797	20
21	19236·797	673·268	680·234	20590·299	21
22	20590·299	720·661	680·234	21991·194	22
23	21991·194	769·692	620·234	23441·120	23
24	23441·120	820·439	680·234	24941·793	24
25	24941·793	872·973	680·234	26495·000	25

Calculation (XV) 1.

Standard Calculation Form, No. 3x.

To find the annual sinking fund instalment to be provided out of revenue or rate to repay the loan under the original conditions laid down at the time of borrowing.

Table III.

Required the annual instalment to be set aside and accumulated as a sinking fund at $3\frac{1}{2}$ per cent. per annum to provide £26,495 in 25 years.

(A) By Formula. $Ay = M\left(\frac{r}{R^N - 1}\right)$ Rule 1, Chapter XIII.

Log. $R^N - 1$: Log. Ratio	R	1·035	0·0149403
Multiply Log. R by	N	25	25
	R^N	$(1·035)^{25}$	0·3735087
Convert Log. to ordinary number	R^N	2·36324	
deduct unity	−1	1·	
Log. of this is	$R^N - 1$	1·36324	0·1345738
Log. Amount of Loan	M	26,495	4·4231639
add Log. r	r	0·035	$\bar{2}$·5440680
	M r		2·9672319
deduct Log. $(R^N - 1)$ above	$R^N - 1$		0·1345738
	Ay		2·8326581

Required annual instalment, £680·2336.

(B) By Table III. $Ay = \dfrac{M}{\frac{R^N - 1}{r}}$ Rule 2, Chapter XIII.

Log. Amount of Loan	M	26,495	4·4231639
Table III. 25 years, $3\frac{1}{2}$ per cent. Amount of £1 per annum *deduct* Log.	$\frac{R^N - 1}{r}$	38·94986	1·5905058
	Ay		2·8326581

Required annual instalment, £680·2336.

(C) By Thoman's Table. $Ay = M\left(\frac{a_n}{R^N}\right)$ Rule 3, Chapter XIII.

$3\frac{1}{2}$ per cent., 25 years.

Log. Amount of Loan	M	26,495	4·4231639
add Log. a^n	a^n		8·7830029
	M a^n		13·2061668
deduct Log. R^N in Table + 10	R^N		10·3735087
	Ay		2·8326581

Required annual instalment, £680·2336.

Calculation (XV) 2.

Standard Calculation Form, No. 3.

To find the amount which should stand to the credit of a sinking fund at any time.

Required the amount which should stand to the credit of a sinking fund representing the accumulation of an annual instalment of £680·234 for 12 years at $3\frac{1}{2}$ per cent.

(A) By Formula. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 1, Chapter VI.

Log $R^N - 1$	Log. Ratio *Multiply* Log. R by	R N	1·035 12	0·0149403 12
		R^N	$(1·035)^{12}$	0·1792842
	Convert Log. to ordinary number *deduct* unity	R^N -1	1·51107 1·	
	Log. of this is	$R^N - 1$	0·51107	$\bar{1}$·7084792
	Log. Annuity *add* Log. $R^N - 1$ above	Ay $R^N - 1$	680·234	2·8326581 $\bar{1}$·7084792
	deduct Log. r	$Ay(R^N - 1)$ r	0·035	2·5411373 $\bar{2}$·5440680
		M		3·9970693

Required amount, £9932·744.

(B) By Table III. $M = Ay\left(\frac{R^N - 1}{r}\right)$ Rule 2, Chapter VI.

Table III. 12 years, $3\frac{1}{2}$ per cent. Amount of £1 per annum	$\frac{R^N - 1}{r}$	14·60196	1·1644112
add Log. Annuity	Ay	680·234	2·8326581
	M		3·9970693

Required amount, £9932·744.

(C) By Thoman's Table. $M = Ay\left(\frac{R^N}{a^n}\right)$ Rule 3, Chapter VI.

$3\frac{1}{2}$ per cent., 12 years.

Log. Annuity *add* Log. R^N in Table + 10	Ay R^N	680·234	2·8326581 10·1792842
deduct Log. a^n	$Ay\, R^N$ a^n		13·0119423 9·0148730
	M		3·9970693

Required amount, £9932·744.

CHAPTER XVI.

SINKING FUND PROBLEMS RELATING TO THE AMOUNT IN THE FUND.

The correction of a deficiency in the fund.

Variation I.

By an additional annual instalment to be set aside during the whole of the unexpired portion of the repayment period. Statement XVI. A.

Variation II.

By an additional annual instalment to be set aside during the earlier part only of the unexpired portion of the repayment period. Statement XVI. C.

Summary of the methods of adjusting a deficiency. The several methods described. The annual increment (balance of loan) method. Statement showing the final repayment of the loan by the operation of the amended annual instalment, in each of the above variations.

Summary of the methods of adjustment.

Variation I (Deficiency), *in which the adjustment is made by an additional annual instalment to be set aside during the whole of the unexpired portion of the original repayment period.*
Statement XVI. A.

(1) *Ascertain the amount of the present deficiency and calculate the equivalent amount of original loan by one of the methods described in Chapter XV.*
Calculation (XV) 6. £734·659.

(2) *Calculate the additional annual sinking fund instalment to be set aside and accumulated for the whole of the unexpired portion of the original repayment period to provide the above equivalent amount of original Loan.*
Calculations (XV) 3 and (XVI) 1. £45·594.

(3) *The additional annual instalment so ascertained added to the original annual instalment will give the augmented annual instalment to be set aside during the whole of the unexpired portion of the repayment period.*

(4) *Prepare a statement showing the final repayment of the loan by the operation of the fund under the amended conditions.* *Statement XVI. A.*

(5) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired portion of the repayment period for after reference.* *Pro forma Account, No. 2.*

VARIATION II (DEFICIENCY), *in which the adjustment is made by an additional annual instalment (£104·039) to be set aside during part only (5 years) of the unexpired portion (13 years) of the original repayment period (25 years).*

Statement XVI. C.

Note. In order to make the following summary perfectly clear it contains (in brackets) the results ascertained in the example afterwards worked out in detail. The annual increment (ratio) method, previously referred to in Chapter XIV, cannot be applied to cases in which the amended instalment is not spread equally over the whole of the period.

(1) *Ascertain the amount of the present deficiency (£469·744) and calculate the equivalent amount (£734·659) of original loan, as described in Chapter XV.*

Calculation (XV) 6.

(2) *Divide the unexpired portion (13 years) of the original repayment period (25 years) into two parts, as follows:—*

1st portion (5 years), during which the additional annual instalment is required to be set aside.

2nd portion (8 years), during which the additional annual instalment is not required to be set aside, but only the annual instalment as originally ascertained.

(3) *Calculate the present value (£557·908) of the above equivalent amount (£734·659) of the original loan, as if it were due at the end of a number of years (8) equal to the second portion of the unexpired repayment period (13 years).* *Calculation (XVI) 3.*

(4) *Calculate the additional annual instalment (£104·039) to be set aside and accumulated for a number of years (5) in the first portion of the unexpired repayment period (13 years) to provide the present value (£557·908) so found, as above.* *Calculation (XVI) 4.*

(5) *The additional annual instalment so found (£104·039) added to the original annual instalment (£680·234) will give the augmented annual instalment (£784·273) to be set aside during the first portion (5 years) of the unexpired repayment period (13 years).*

(6) *The original annual instalment (£680·234) will continue to be set aside and accumulated during the second portion (8 years) of the unexpired repayment period (13 years).*

(7) *Prepare a statement showing the final repayment of the loan by the operation of the fund under the amended conditions.* *Statement XVI. C.*

(8) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired period for reference in after years.*
Pro forma Account, No. 3.

Note. The calculations in this and subsequent chapters will be found in the Appendix, but each calculation will be shown by only one of the three methods given in the standard forms.

Variation I. The correction of a deficiency in the fund by an additional annual instalment to be set aside during the whole of the unexpired portion of the repayment period.
Statement XVI. A.

In Chapter XV, the factors relating to a deficiency in a sinking fund have been fully discussed, and several methods described by which to ascertain the resulting additional annual instalment to be spread equally over the whole of the unexpired portion of the repayment period. Two alternative methods have been pointed out by which the deficiency may be corrected, both of which agree in providing an additional annual instalment, but differ as to the number of years over which such increased contributions shall be spread. Sound finance demands that the error should be put right by an immediate payment of the deficiency into the fund, or that the increased annual contribution should be spread over a shorter term than the full unexpired portion of the original repayment period, but the

circumstances of individual cases may render it more equitable, or perhaps more convenient, that the adjustment should be spread over the longest possible period.

The present deficiency of £469·744 if immediately paid into the fund and accumulated until the end of the period, will then provide £734·659 of original loan which would otherwise have been unprovided for. The additional annual instalment of £45·594, to be set aside and added to the fund during the whole of the unexpired period, has already been ascertained by Calculation (XV) 3, and it will now be proved by a further Calculation (XVI) 1 upon the author's standard calculation form No. 3x, based upon Table III, which is the usual method of finding the sinking fund instalment. This and other calculations subsequently referred to will be found in the appendix.

Having ascertained the required additional annual instalment, it is now possible to review the operation of the fund so amended in order to show the final repayment of the loan by the following Statement XVI. A., which will apply equally to all similar cases of adjustment due to a deficiency in the fund, irrespective of the method by which the additional annual instalment is ascertained provided that such additional annual instalment be spread equally over the whole of the unexpired portion of the repayment period.

The following Statement XVI. A. also shows that the present investments of £9,463 will, if accumulated at $3\frac{1}{2}$ per cent. until the end of the period, then provide for the repayment of £14799·71 of original loan. Before making the above correction the balance of the loan unprovided for was represented by:—

The remaining original annual instalments of £680·234 and their accumulations Calculation (XV) 5		£10960·62
The deficiency at the end of the 12th year	£469·74	
and the loss of accumulated interest caused thereby	264·93	
		£734·67
Balance of Loan		£11695·29

After making the above adjustment this amount will be provided by the accumulation of the augmented annual instalment of £725·828, as shown by Calculation (XVI) 2.

THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. The annual increment has been fully described in Chapter XIV, where it is shown that it may be used to simplify the majority of the adjustments in a sinking fund, rendered necessary by any variation from the original conditions as to the repayment of the loan. There is, however, one limitation, namely, that any variation in the rate of income to be received upon the present investments, or in the rate of accumulation, must apply equally to the whole of the future period of repayment, which, however, may be increased or reduced. It is also necessary that any increased or reduced annual instalment, consequent upon any such variation in the original conditions, shall be spread equally over the whole of the unexpired portion of the repayment period. For this reason, therefore, the method has been applied in Statement XVI. B. to the foregoing example (Variation I) in which the deficiency of £469·744 is made good by an additional annual instalment of £45·594, to be spread equally over the whole of the unexpired portion of the repayment period, but the method will not apply to the example following, namely, Variation II, in which the additional annual instalment is required to be spread over the earlier years only of such unexpired term. If this method be applied to the latter example the result would be only the *equated* annual instalment, which, however interesting from a theoretical point of view, would not be of any practical use under the actual conditions. An example of an equated annuity is given and fully described in Chapter XXVII.

This method of making the adjustment of a sinking fund by means of the annual increment is practically the same as that adopted in the case of local authorities, where the whole of the annual instalments, as and when set aside, are immediately applied in the actual repayment of debt. Section 234 (5) of the Public Health Act, 1875, provides that where any part of the fund is so applied there shall be paid into the fund and charged to the rate account the interest which would have been earned by the part of the fund so applied. If it be assumed that the whole of the fund is so applied in repayment of the debt, and the rate of interest payable upon the loan is the same as the rate of accumulation of the fund, the amount charged annually to rate account in respect of interest and redemption charges, is the annual increment of the fund, using the term in the sense here applied to it.

A Deficiency in the Fund. **Statement XVI. A.**

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the sinking fund, after making the adjustment in the annual instalment, consequent upon a deficiency in the amount which should stand to the credit of the fund.

VARIATION I (DEFICIENCY), in which the additional annual instalment is set aside during the whole of the unexpired portion of the repayment period.

		Equivalent amount of original loan.
Present investments (at end of 12th year),	£9463·00	
Amount thereof, accumulated for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 4		£14799·71
Amended annual instalment :—		
Original annual instalment	£680·234	
Additional annual instalment Calculation (XVI) 1	45·594	
	£725·828	
Amount thereof in 13 years at $3\frac{1}{2}$ per cent. Calculation (XVI) 2		£11695·29
Amount of original loan		£26495·00

A Deficiency in the Fund. Statement XVI. B.

The Annual Increment (balance of loan) Method.

To find the amended annual sinking fund instalment consequent upon a deficiency in the amount which should stand to the credit of the fund.

VARIATION I (DEFICIENCY), in which the additional annual instalment is set aside during the whole of the unexpired portion of the repayment period.

Amount of original loan (25 years)		£26495·00
deduct amount in the fund at the end of the 12th year		£9463·00
Balance of loan		£17032·00
Amended annual increment to be added to the fund and accumulated at $3\frac{1}{2}$ per cent., to provide this amount at the end of 13 years. Calculation (XVI) 9		£1057·033
deduct income to be received from the present investments, £9,463, at $3\frac{1}{2}$ per cent.		£331·205
Amended annual instalment, *being*		£725·828
Original annual instalment ...	£680·234	
Additional annual instalment	£45·594	
		£725·828

The rule relating to this method is stated at the head of Chapter XXII.

Pro forma Sinking Fund Account, No. 2.

A Deficiency in the Fund. (Variation I.)

Loan of £26,495, repayable at the end of 25 years.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the amended annual instalment of £725·828, to be set aside during the whole of the unexpired period of repayment.

Statement XVI. A. Rate of accumulation, $3\frac{1}{2}$ per cent.

The amount in the fund at the end of the 12th year, £9,463, is an assumed amount, and is equivalent to setting aside an annual instalment of £648·064, as shown by Calculation (XVI) 10, instead of the correct annual instalment of £680·234.

Year.	Amount in the fund at beginning of year.	Income received from investments $3\frac{1}{2}$ per cent	Annual sinking fund instalment.	Amount in the fund at end of year.	Year.
1					1
2					2
3					3
4					4
5					5
6					6
7					7
8					8
9					9
10					10
11					11
12				9463·000	12
13	9463·000	331·205	725·828	10520·033	13
14	10520·033	368·201	725·828	11614·062	14
15	11614·062	406·492	725·828	12746·382	15
16	12746·382	446·123	725·828	13918·333	16
17	13918·333	487·142	725·828	15131·303	17
18	15131·303	529·596	725·828	16386·727	18
19	16386·727	573·535	725·828	17686·090	19
20	17686·090	619·013	725·828	19030·931	20
21	19030·931	666·083	725·828	20442·842	21
22	20442·842	714·799	725·828	21863·469	22
23	21863·469	765·221	725·828	23354·518	23
24	23354·518	817·408	725·828	24897·754	24
25	24897·754	871·418	725·828	26495·000	25

VARIATION II. The correction of a deficiency in the fund by an additional annual instalment, to be set aside during the earlier years only of the unexpired portion of the repayment period. Statement XVI. C.

The correction of the deficiency in this manner is more complicated than by spreading the additional annual instalment equally over the whole of the unexpired portion of the repayment period, but is not at all difficult. The factors immediately concerned are (1) the present deficiency of £469·744; (2) the amount of original loan £734·659, represented by such deficiency, and (3) the original annual instalment of £680·234.

In the present example it will be assumed that the additional annual instalment is required to be of such increased amount (as compared with the additional annual instalment of £45·594 to be spread over the whole of the unexpired period) that it will be sufficient to make up the present deficiency if set aside for 5 years only, instead of for 13 years. Under this alternative method the unexpired period of 13 years is divided into two parts. During the first five years the additional annual instalment will be set aside and accumulated at $3\frac{1}{2}$ per cent. in augmentation of the original annual instalment. At the end of the five years this additional annual instalment will cease, and will then have amounted to a sum which will continue to accumulate at compound interest for a further eight years. The accumulated amount of the additional annual instalment at the end of five years, should, at the end of the remaining eight years, amount to the balance (£734·659) of loan not otherwise provided for. The adjustment may be made by direct calculation, and may also be made by steps. A similar method by step has been adopted when dealing with a variation in the future rate of income to be received upon the present investments when it is known in advance that such a variation will take effect at a definite future date during the unexpired portion of the redemption period, as explained in Chapter XXVII. In order to determine the additional annual instalment to be set aside and accumulated for the first period of five years, it is first necessary to ascertain the sum to which it is required to accumulate at the end of five years, which latter sum will in its turn accumulate without further addition for a further period of eight years. At the end of the unexpired period of 13 years it is necessary to provide £734·659, and the first step is to ascertain the sum which, if accumulated at $3\frac{1}{2}$ per cent. for eight years, will amount to £734·659; in other words, to find the present value of £734·659 under the above conditions, namely,

£557·908. Calculation (XVI) 3. The next step is to ascertain the annual instalment which will amount to £557·908 if set aside and accumulated at $3\frac{1}{2}$ per cent. for five years.

This is a similar problem to the previous one dealing with the present deficiency of £469·744, where it was required to find the annual instalment to amount to £734·659, Calculation (XVI) 1, and also similar to Calculation (XV) 1 required to find the original annual instalment of £680·234. Calculation (XVI) 4 shows that the equal annual instalment to provide £557·908 at the end of five years at $3\frac{1}{2}$ per cent. is £104·039. The method of complying with the above conditions has now been ascertained. It has been found by Calculation (XVI) 4 that an annual instalment of £104·039 set aside for five years and accumulated at $3\frac{1}{2}$ per cent. will at the end of that time amount to £557·908, and it has been found by Calculation (XVI) 3 that this sum of £557·908, accumulated at $3\frac{1}{2}$ per cent. for eight years, will amount to £734·659, which is the portion of the original loan not otherwise provided for, owing to the present deficiency of £469·744.

The sinking fund, as amended by the results of the foregoing calculations will now consist of:—

A present credit to the fund, represented by investments valued at		£9463·000
An augmented annual instalment for 5 years made up as follows:		
Original instalment	£680·234	
Additional instalment for 5 years	104·039	
		£784·273
The original annual instalment to be continued for a further 8 years of		£680·234

And the above provision accumulated at $3\frac{1}{2}$ per cent., as originally calculated, will at the end of the prescribed period of repayment, namely, 25 years, be sufficient to provide the full amount of the original loan of £26,495.

In order to complete the argument it is necessary to show the position of the fund at the end of the 17th year when the additional annual instalment of £104·039 will cease and to continue the accumulation of the fund from that time until the end of the original term of 25 years. During the second period of eight years, as previously mentioned, the original instalment

of £680·234 only will continue to be set aside and added to the fund. The following Statement XVI. C. shows the final repayment of the loan by the operation of the fund after making the above adjustment.

In the foregoing statement a break has been made at the end of the 17th year, being the end of the five years during which the corrective instalment of £104·039 is required to be set aside. The calculation might have been simplified by ascertaining, in the direct manner shown in Statement XVI. D.1, the amount of loan which will be provided by the accumulation at the end of 13 years of the instalment of £104·039 to be set aside for five years only. This direct method by step is fully explained in Chapter XXVII, Statement C., where it is applied to find the amount of loan which will be provided by the accumulation of the income from the present investments, such income being at varying known rates per cent. during the unexpired period. (The calculation might also have been made in terms of the amended annual instalment of £784·273). In conclusion, a further Statement XVI. D.2, has been prepared, showing the final repayment of the loan, which should be compared with Statement XVI. C., in order to show the simplification of the proof by the method by step.

A Deficiency in the Fund. **Statement XVI. C.**

Showing the final repayment of the loan, by the operation of the sinking fund after making the adjustment in the annual instalment consequent upon a deficiency in the amount which should stand to the credit of the fund.

Variation II (Deficiency), in which the additional annual instalment is set aside during the earlier part only of the unexpired portion of the repayment period.

		Equivalent amount of original loan.
Present investments (at end of 12th year),	£9463·00	
Amount thereof, accumulated for 5 years at 3½ per cent. Calculation (XVI) 5		£11239·07
Amended annual instalment :—		
Original	£680·234	
Additional	104·039	
	£784·273	
Amount thereof, accumulated for 5 years at 3½ per cent. Calculation (XVI) 6		£4205·64
Amount in the fund, at end of 17th year		£15444·71
Amount thereof, accumulated for 8 years at 3½ per cent. Calculation (XVI) 7		£20337·74
Original annual instalment (resumed) :—		
Amount of £680·234 per annum, accumulated for 8 years at 3½ per cent. Calculation (XVI) 8		£6157·26
Amount of original loan		£26495·00

A Deficiency in the Fund. Statement XVI. D (1).

The Amount of (the Amount of £1 per Annum) Method by Step, by Thoman's Tables.

To find the accumulated amount of an additional annual instalment, or other annuity, to be set aside and added to the sinking fund for a limited period of years; and at the end of that period the accumulated amount thereof to continue to accumulate for a further specified period. The rate of accumulation in both periods may be the same, or be at different rates per cent.

Required the amount of an additional annual instalment of £104·039, to be set aside for a period of 5 years, and accumulated at 3½ per cent. At the end of 5 years the annual instalment ceases, but the sum to which it has then amounted continues to accumulate for a further period of 8 years, also at 3½ per cent.

First period, 5 years. Second period, 8 years.

Log. instalment		*Ay*	104·039	2·0171984
add: Log. R^N, 3½ per cent.	5 years	R^N		0·0747017
Log. R^N, 3½ per cent.	8 years	R^N		0·1195228
				2·2114229
add 10 to the log.				12·2114229
deduct: Log. a^n, 3½ per cent.	5 years	a^n		9·3453372
		M		2·8660857
which is the log. of the required future amount, namely				£734·659

Note. This method may be inverted to find the additional annual instalment in the first instance instead of as described in the text. See Statement XXXIV. G.

A Deficiency in the Fund. Statement XVI. D (2).

Showing the final repayment of the loan, by the operation of the sinking fund after making the adjustment in the annual instalment, consequent upon a deficiency in the amount which should stand to the credit of the fund.

Variation II (Deficiency), in which the additional annual instalment is set aside during the earlier part only of the unexpired portion of the repayment period.

An alternative method to Statement XVI. C., based upon the method by step.

		Equivalent amount of original loan.
Present investments (at end of 12th year),	£9463·00	
Amount thereof, accumulated for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 4		£14799·71
Original annual instalment	£680·234	
Amount thereof, accumulated for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 5		£10960·62
Additional annual instalment	£104·039	
to be set aside for 5 years only, and accumulated for a further 8 years at $3\frac{1}{2}$ per cent "Method by step" Calculation (XVI) D.1		£734·67
Amount of original loan		£26495·00

Pro forma Sinking Fund Account, No. 3.

A Deficiency in the Fund. (Variation II.)

Loan of £26,495, repayable at the end of 25 years.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the amended annual instalment of £784·273, to be set aside during the first 5 years only of the unexpired period of repayment.

Statement XVI. C. Rate of accumulation, $3\frac{1}{2}$ per cent.

Year.	Amount in the fund at beginning of year.	Income received from investments $3\frac{1}{2}$ per cent.	Annual sinking fund instalment.	Amount in the fund at end of year.	Year.
1					1
2					2
3					3
4	The amount in the fund at the end of the				4
5	12th year, £9,463, is an assumed amount,				5
6	and is equivalent to setting aside an				6
7	annual instalment of £648·064, as shown				7
8	by Calculation (XVI) 10, instead of the				8
9	correct annual instalment of £680·234.				9
10					10
11					11
12				9463·000	12
13	9463·000	331·205	784·273	10578·478	13
14	10578·478	370·247	784·273	11732·998	14
15	11732·998	410·655	784·273	12927·926	15
16	12927·926	452·477	784·273	14164·676	16
17	14164·676	495·761	784·273	15444·710	17
18	15444·710	540·567	680·234	16665·511	18
19	16665·511	583·293	680·234	17929·038	19
20	17929·038	627·516	680·234	19236·788	20
21	19236·788	673·288	680·234	20590·310	21
22	20590·310	720·661	680·234	21991·205	22
23	21991·205	769·692	680·234	23441·131	23
24	23441·131	820·440	680·234	24941·805	24
25	24941·805	872·961	680·234	26495·000	25

CHAPTER XVII.

SINKING FUND PROBLEMS RELATING TO THE AMOUNT IN THE FUND.

A surplus in the fund; how it may arise, and how it may be adjusted.

Variation I.

ARISING IN CONSEQUENCE OF AN EXCESSIVE PAST ACCUMULATION OF THE FUND.

Variation II.

ARISING IN CONSEQUENCE OF THE PAYMENT INTO THE FUND OF THE PROCEEDS OF SALE OF PART OF THE ASSETS REPRESENTING THE SECURITY FOR THE LOAN, OR A REALISED PROFIT UPON THE SALE OF AN INVESTMENT REPRESENTING THE FUND.

STATEMENT XVII. A.

SUMMARY OF THE METHODS OF ADJUSTMENT. THE VARIOUS CAUSES LEADING TO A SURPLUS IN THE FUND. DIFFERENCE IN CONDITIONS AND PRACTICE AS BETWEEN LOCAL AUTHORITIES AND COMMERCIAL AND FINANCIAL UNDERTAKINGS. COMPARISON OF THE VARIOUS METHODS OF DEALING WITH A SURPLUS. THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN BY THE OPERATION OF THE AMENDED ANNUAL INSTALMENT.

Summary of the methods of adjustment.

VARIATION I (SURPLUS), *arising in consequence of an excessive past accumulation of the fund.*

(1) *Ascertain the actual present surplus, as described in Chapter XV.*

(2) *Calculate the annuity or annual instalment of which this sum is the present value for the unexpired portion of the repayment period.* *Similar to Calculation (XV) 3.*

(3) *The annual instalment so ascertained, deducted from the original annual instalment will give the reduced annual instalment to be set aside during the whole of the unexpired portion of the repayment period.*

(4) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions. Similar to Statement XVI. A., with the necessary modifications relating to a surplus instead of to a deficiency.*

(5) *Prepare a pro forma amount showing the amount which should be in the fund at the end of each year of the unexpired repayment period.*

Note. The above method so closely resembles the one adopted in the case of a deficiency and the following method relating to Variation II, that no further amplification is required. Unlike a deficiency, however, a surplus should be spread equally over the whole of the unexpired repayment period, and consequently Method II (Deficiency) will rarely apply.

VARIATION II (SURPLUS), *arising in consequence of the payment into the fund of the proceeds of sale of part of the assets representing the security for the loan, or a realised profit upon the sale of an investment representing the fund.*

Statement XVII. A.

(1) *Ascertain in the manner described in Chapter XV, whether there is a surplus or a deficiency in the fund apart from the proceeds of realisation now under consideration; and if so, calculate the corrective annual sinking fund instalment required. Calculation (XV) 3.*

(2) *Calculate the annuity which may be purchased for the unexpired portion of the repayment period, with the amount now paid into the fund. Calculation (XVII) 1.*
[Here refer to memo. after (6).]

(3) *Deduct the annuity so ascertained from the original annual instalment, and adjust the latter also, if required, by the above corrective instalment, referred to in (1).*

(4) *The remainder will be the future reduced annual sinking fund instalment, to be set aside and accumulated during the whole of the unexpired portion of the repayment period.*

(5) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XVII. A.*

(6) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired repayment period.* *Pro forma Account, No. 4.*

Memo. If the original annual instalment be a prescribed sum instead of being found by calculation in the ordinary way (see (XVI) 1), proceed by the method described under Variation IV (Surplus) in Chapter XVIII, substituting for operation (7) in that method the above operation (2).

A Surplus in a Sinking Fund and how it may arise.

Although it does not fall within the province of a work of this character to mention all the various causes which may lead to the existence of a surplus in a sinking fund, yet it is very advisable to give a brief outline of the principal ways in which this may happen, and which may be divided into the following classes, any two or more of which may operate simultaneously:—

(1) An excess in the amount of the annual instalments previously paid into the fund or an increase in the rate of accumulation in excess of the rate assumed in calculating the original instalment. Variation I.

(2) The payment into the fund of a realised profit upon the sale of an investment representing the fund or the proceeds of sale of part of the assets representing the security for the loan. Variation II.

(3) In the case of commercial or financial undertakings, there may be a change in the character of part of the original loan, whereby the original obligation to set aside a sinking fund is modified owing to the withdrawal of part of the loan from the operation of the fund.

This will be fully discussed in the following chapter, where it will be shown that the precise method of making the adjustment depends upon the nature of the original annual instal-

ment, and the problem will be divided into two parts as follows:—

A. In which the original annual instalment was found by calculation based upon a specified period of repayment and rate of accumulation. Variation III.

B. In which the original annual instalment is a stated sum and is not based, except in a general way, upon any period of repayment or rate of accumulation. Variation IV.

VARIATION I (SURPLUS), arising in consequence of an excessive past accumulation of the fund.

This will be of rare occurrence if the pro forma account already recommended has been made out showing the operation of the fund until maturity, and any such minor instances may be adjusted as and when they arise by transfers to the current year's rate or revenue account. In the case of larger amounts they may be treated in the manner mentioned in Chapters XV and XVI, referring to a deficiency in the fund, but of course by reducing the future annual instalment.

VARIATION II (SURPLUS), arising in consequence of the payment into the fund of the proceeds of sale of part of the assets representing the security for the loan or a realised profit upon the sale of an investment representing the fund. Statement XVII. A.

This chapter will deal fully with those cases in which the sinking fund obligations are modified by the payment into the fund of the proceeds of sale of part of the security for the loan to be ultimately repaid, and attention will be directed to the difference in practice as between local authorities and commercial and financial undertakings. In the case of a local authority the sinking fund instalment is set aside to repay the loan at the end of the period allowed under the general or special Act. These loans are invariably expended upon works of a capital nature, and it sometimes happens that part of the property representing the security for the loan is sold. The practice generally followed in the case of local authorities is to pay such proceeds into the fund and apply the same in the redemption or repayment of part of the original loan. This is as it should be, and is the practice adopted in commercial and financial undertakings, but it has an important effect upon the sinking fund instalment. The repayment, during the period

allowed, of part of the debt out of the proceeds of sale of part of the security (instead of out of the sinking fund provided out of current rates or profits) anticipates the natural effect of the sinking fund, and by reducing the loan repayable at the end of the period correspondingly reduces the necessity to set aside in future the full original sinking fund instalment.

It is obvious therefore that the original annual sinking fund instalment may be reduced during the remainder of the term to such an amount as will provide the balance of the debt not repaid out of the proceeds of the sale of part of the assets.

This principle is followed in the case of commercial and financial undertakings, but in the case of local authorities the Local Government Board may require that the proceeds of such sales shall be paid into the sinking fund, and that the full amount of the original annual instalment shall continue to be set aside. The effect of this is to shorten the original period allowed for the redemption of the debt. There is not any objection to this method except that the result is to relieve the later generation of ratepayers at the expense of the present, but in its favour is the fact that it is always sound finance to repay debt as soon as possible. In the case of commercial and financial undertakings the practice varies, depending in each instance upon the conditions laid down in the deed relating to the loan. Generally speaking, it may be considered equitable in the case of such undertakings to reduce the sinking fund instalment and so maintain the original period allowed for the repayment of the debt. In the case of a debenture stock repayable on a fixed future date this would necessarily require to be so unless part of the stock were redeemed by purchase upon the open market.

The proceeds of sales of capital assets forming part of the security would, failing actual redemption, be invested in securities authorised by the deed, and the resulting income would be added to the sinking fund during the unexpired portion of the repayment period, and therefore the future annual instalments to be provided out of the profits of the undertaking would be correspondingly reduced. If, however, in the case of a commercial or financial undertaking any such proceeds arising from the sale of part of the security were actually applied in redemption of part of the loan, instead of being invested in outside securities, the profit and loss account of the undertaking would be relieved to the extent of the annual interest payable upon such redeemed debt, but the sinking fund would not then be increased by any income arising from the

investment. Another difference between the sinking funds of commercial and financial undertakings and those of local authorities arises from the fact that in the former the annual instalment is not always charged against the profits of the undertaking but may be taken out of the general assets of the concern.

The method of adjustment will be illustrated by the following example relating to a commercial or financial undertaking.

A sinking fund has been set aside and accumulated to provide for the repayment of a loan of £26,495 at the end of 25 years—and in fixing the annual instalment the rate of accumulation was taken at $3\frac{1}{2}$ per cent. At the end of the 12th year the fund stands at the proper estimated amount shown by the pro forma account, and as found by Calculation (XV) 2, namely, £9932·744. At that time a portion of the assets (forming part of the security for the loan) is realised, and produces, say, £4,560.

The trust deed provides that this amount shall be paid into the sinking fund and invested, and accumulated until the loan is repayable, namely, at the end of the 25th year, and that the future annual sinking fund instalments may be correspondingly reduced. In the present example there is not any question of the rate of income on the present investments, or the future rate of accumulation, being less than $3\frac{1}{2}$ per cent., the rate originally assumed in calculating the annual instalment. The effect of the realisation of part of the security for the loan is that the amount in the sinking fund is suddenly increased by the sum of £4,560, which was not anticipated when the original annual instalment was calculated. If therefore this amount be paid into the sinking fund and accumulated, and the original instalments continue to be set aside in future and paid into the fund until the end of the 25th year, the sinking fund will at the end of that period be in excess of the amount required to repay the loan, and the excess will be the amount of the above sum of £4,560 accumulated at $3\frac{1}{2}$ per cent., compound interest, for 13 years.

The method of ascertaining the amount by which the original annual instalment may be reduced during the unexpired portion of the repayment period is exactly similar in principle to that adopted in the case of the deficiency of £469·744 in Chapter XV. In that case the deficiency was converted into terms of original loan and the annual instalment to be set aside during the remaining 13 years to redeem the portion of the loan not already provided for was ascertained.

In the present case the sinking fund stands at the proper calculated amount at the end of the 12th year; and the original annual instalments, alone, if continued for a further 13 years, will be amply sufficient to provide for the ultimate repayment of the debt. In addition there is a sum in hand of £4,560, which may now be applied in repaying part of the loan, and the equivalent annuity for the remainder of the period may be applied in reduction of the future annual instalments to be added to the fund. The £4,560 may be regarded as a sum which may now be invested in the purchase of an annuity for 13 years on a $3\frac{1}{2}$ per cent. basis. This method is the more preferable seeing that the £4,560 is actually in hand, whereas the deficiency of £469·744 represented the present value of a sum due at a future period and was a definite amount only so far as it represented a sum which should have been in actual possession, but was not so in fact.

When discussing the adjustment of a sinking fund in the case of a deficiency in Chapters XV and XVI several alternative methods were pointed out depending upon the period allowed in which to make good past deficiencies. In the case of the surplus under review, there is not any alternative to that already considered if the original date of repayment be adhered to, because the sum in question is a definite one and is actually in hand. The calculation will be made upon the author's standard form No. 5, relating to the annuity which £1 will purchase. It will be seen that the sum now paid into the fund will effect a decrease in the original annual instalment of £442·6008 per annum. Calculation (XVII) 1.

The final repayment of the loan by the operation of the amended instalment during the remaining 13 years of the original repayment period is shown in the following Statement XVII. A.

The above method should be carefully compared with the correction of a surplus in a sinking fund, caused by the withdrawal of part of the loan from the operation of the fund owing to the conversion of such part of the loan into ordinary share capital. The difference in the methods will be fully described in Chapter XVIII.

THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. This method will now be used for the purpose of ascertaining the amended annual instalment, based upon the future annual increment, a summary of which is given at the beginning of

Chapter XV, and is fully described in Chapters XVI and XXII. As this method is based upon the same actual conditions as the previous example, Statement XVII. A., showing the final repayment of the loan, will also apply. This method is shown in Statement XVII. B.

Comparison of the Methods of Dealing with a Surplus and a Deficiency in a Sinking Fund.

It is instructive to compare the above results with the example worked out in the case of a deficiency in the fund (Variation I), seeing that both funds relate to loans identical as to amount, period of repayment, and rates per cent. of income and accumulation.

In each case also the adjustment is made at the end of the 12th year, and is spread over the full remaining term of 13 years.

In the case of the deficiency in the fund there was an ascertained amount of £469·744 by which the present investments, £9463·00, fell short of the amount of £9932·744 which should have been in the fund in order to carry out the original obligation. This deficiency was corrected by setting aside an additional annual instalment, in augmentation of the original annual instalment of £680·234 during the unexpired portion of the repayment period.

This instalment of £45·594, which was found by Calculation (XV) 3, represents the annuity which might have been purchased with the above amount of £469·744.

The present surplus consists of an actual amount of cash, namely, £4,560, paid into the fund, which is applied in providing an instalment in reduction of the original annual instalment of £680·234. This instalment, as found by Calculation (XVII) 1, based on Table V, is £442·601, and represents the annuity which might be purchased with the above amount of £4,560 paid into the fund.

In both cases the amount, which should, as shown by Calculation (XV) 2, have been to the credit of the fund, is £9932·744, which amount, if accumulated for 13 years at $3\frac{1}{2}$ per cent., would at the end of the period, as shown by Calculation (XVII) 2, have amounted to £15534·375 of original loan.

In the case of the surplus caused by a payment into the fund now under consideration, part of the amount which should be in the fund at the end of the prescribed repayment period of 25 years is actually in hand at the end of the 12th year, and

therefore the future annual instalment must be correspondingly reduced owing to the future accumulation of the sum of £4,560 paid into the fund.

In the two cases the amount which should have been to the credit of the fund at the end of the 12th year was represented as follows:—

In the case of a Deficiency in the Fund.

	Amount at end of year.	Equivalent amount of loan.
Actual amount in the fund	£9463·	£14799·71
Deficiency, involving an additional annual instalment of £45·594 ...	469·744	734·67
	£9932·744	£15534·38

In the case of a Surplus in the Fund.

	Amount at end of year.	Equivalent amount of loan.
Actual amount in the fund	£9932·744	£15534·38
Deficiency. Nil.	—	—
	£9932·744	£15534·38

In both cases the amount of original loan to be provided by the accumulation of the future annual instalments for 13 years is the same, namely, £10960·62, being the total of the original loan, £26,495, after deducting the above amount of £15534·38 already provided for.

The manner in which this remaining portion of original loan is dealt with in the two cases is shown in the following table:—

In the case of a Deficiency in the Fund.

	Equivalent amount of loan
Future original annual instalment of £680·234, to be set aside and accumulated for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 5	£10960·62

A Surplus in the Fund. **Statement XVII. A.**

Showing the final repayment of the loan, by the operation of the sinking fund after making the adjustment in the annual instalment consequent upon a surplus over the amount which should stand to the credit of the fund.

Variation II (Surplus), arising in consequence of the payment into the fund of the proceeds of sale of part of the assets representing the security for the loan, etc.

		Equivalent amount of original loan.
Present investments (at end of 12th year),	£9932·74	
Amount thereof, accumulated for 13 years at 3½ per cent. Calculation (XVII) 2		£15534·38
Amount paid into the fund	£4560·00	
Amount thereof, accumulated for 13 years at 3½ per cent. Calculation (XVII) 3		£7131·64
Amended annual instalment:—		
Original instalment	£680·234	
reduced by, (XVII) 4,	442·601	
	£237·633	
Amount thereof, accumulated for 13 years at 3½ per cent. Calculation (XVII) 5		£3828·98
Amount of original loan		£26495·00

A Surplus in the Fund. Statement XVII. B.

The Annual Increment (balance of loan) Method.

To find the amended annual sinking fund instalment consequent upon a surplus over the amount which should stand to the credit of the fund.

VARIATION II (SURPLUS), arising in consequence of the payment into the fund of the proceeds of sale of part of the assets representing the security for the loan.

Amount of original loan (25 years)		£26495·00
deduct amount in the fund at the end of the 12th year ...	£9932·74	
proceeds of sale paid into the fund	£4560·00	
		£14492·74
Balance of loan		£12002·26

Amended annual increment to be added to the fund and accumulated at $3\frac{1}{2}$ per cent. to provide this amount at the end of 13 years. Calculation (XVII) 6	£744·879
deduct income to be received from the present investments, £14492·74 at $3\frac{1}{2}$ per cent.	£507·246
Amended annual instalment, being:—	£237·633

Original annual instalment	£680·234	
reduced by	£442·601	
		237·633

The rule relating to this method is stated at the head of Chapter XXII.

Pro forma Sinking Fund Account, No. 4.

A Surplus in the Fund. (Variation II.)

Loan of £26,495, repayable at the end of 25 years.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the reduced annual instalment of £237·633.

Statement XVII. A. Rate of accumulation, $3\frac{1}{2}$ per cent.

Year.	Amount in the fund at beginning of year.	Income received from investments $3\frac{1}{2}$ per cent.	Annual sinking fund instalment.	Proceeds of sale of assets paid into the fund.	Amount in the fund at end of year.	Year.
1						1
2						2
3						3
4	The amount in the fund at the end of the 12th year, £9932·744, is the correct calculated amount, as shown by Calculation (XV) 2, and by the pro forma account, No. 1, Chapter XV.					4
5						5
6						6
7						7
8						8
9						9
10						10
11						11
12				4560·000	9932·744	12
13	14492·744	507·246	237·633	—	15237·623	13
14	15237·623	533·317	237·633	—	16008·573	14
15	16008·573	560·300	237·633	—	16806·506	15
16	16806·506	588·228	237·633	—	17632·367	16
17	17632·367	617·133	237·633	—	18487·133	17
18	18487·133	647·050	237·633	—	19371·816	18
19	19371·816	678·014	237·633	—	20287·463	19
20	20287·463	710·061	237·633	—	21235·157	20
21	21235·157	743·230	237·633	—	22216·020	21
22	22216·020	777·561	237·633	—	23231·214	22
23	23231·214	813·092	237·633	—	24281·939	23
24	24281·939	849·868	237·633	—	25369·440	24
25	25369·440	887·927	237·633	—	26495·000	25

CHAPTER XVIII.

SINKING FUND PROBLEMS, RELATING TO THE AMOUNT IN THE FUND.

A surplus in the fund, OF A COMMERCIAL OR FINANCIAL UNDERTAKING ARISING ON THE WITHDRAWAL OF PART OF THE LOAN FROM THE OPERATION OF THE FUND, OWING TO THE CONVERSION OF SUCH PART OF THE LOAN INTO ORDINARY SHARE CAPITAL OR STOCK OF THE UNDERTAKING.

Variation III, IN WHICH THE ORIGINAL ANNUAL INSTALMENT WAS FOUND BY CALCULATION BASED UPON A SPECIFIED PERIOD OF REPAYMENT AND RATE OF ACCUMULATION. STATEMENT XVIII. A.

Variation IV, IN WHICH THE ORIGINAL ANNUAL INSTALMENT IS A STATED SUM AND IS NOT BASED, EXCEPT IN A GENERAL WAY, UPON ANY PERIOD OF REPAYMENT OR RATE OF ACCUMULATION. STATEMENT XVIII. D.

SUMMARY OF THE METHODS OF ADJUSTMENT. REMARKS AS TO THE SINKING FUNDS OF COMMERCIAL AND FINANCIAL UNDERTAKINGS. THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN BY THE OPERATION OF THE AMENDED ANNUAL INSTALMENT.

Summary of the methods of adjustment.

VARIATION III (SURPLUS), *arising on the withdrawal of part of the loan from the operation of the sinking fund of a commercial or financial undertaking owing to the conversion of such part of the loan into ordinary share capital or stock of the undertaking*:—

in which the original annual instalment was found by calculation based upon a specified period of repayment and rate of accumulation. *Statement XVIII. A.*

(1) *Ascertain, in the manner described in Chapter XV, whether there is a surplus or a deficiency in the fund apart from the special circumstances now under review, and if so, calculate the corrective annual sinking fund instalment required, by one of the methods there described.* *Calculations (XV) 3 or (XVI) 1.*

(2) *Calculate the annual sinking fund instalment, which, if set aside for the whole of the unexpired portion of the repayment period, will provide the part of the loan converted into ordinary share capital, and thereby withdrawn from the operation of the fund.*
Calculation (XVIII) 1.

(3) *Deduct the annual instalment so ascertained from the original annual instalment, and adjust the latter if required, by the above corrective instalment.*
Calculation (XVI) 1.

(4) *The remainder will be the future reduced annual instalment, to be set aside and accumulated during the whole of the unexpired portion of the repayment period.*
Calculation (XVIII) 2.

(5) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XVIII. B.*

(6) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired repayment period.* *Pro forma Account, No. 5.*

VARIATION IV (SURPLUS), *arising on the withdrawal of part of the loan from the operation of the sinking fund of a commercial or financial undertaking owing to the conversion of such part of the loan into ordinary share capital or stock of the undertaking:—*

in which the original annual instalment is a stated sum, and is not based, except in a general way, upon any period of repayment or rate of accumulation.
Statement XVIII. D.

(1) *Ascertain from the actual records the value of the present investments representing the fund. Ascertain also the rate of income yielded on such value, and upon this and other considerations, as elsewhere described, base the future rate of accumulation of the fund.*

(2) *Ascertain by inspection of Table III, the approximate number of years in which the stated annual instalment will accumulate to the amount of the original loan at the rate of accumulation fixed as in (1). Adopt the nearest integral number of years so found as the approximate period of repayment of the original loan, at the rate of accumulation, ascertained as above.*

(3) *Calculate the annual sinking fund instalment required to repay the full amount of the loan at the end of the approximate period of repayment found in (2) at the rate of accumulation fixed as in (1).*

Calculation (XVIII) 5. £7441·63.

(4) *Calculate the amount which would be in the fund if the annual instalment (£7441·63) so found (3) had been set aside and accumulated at the rate per cent. fixed in (1) from the date of issue of the loan until the date of conversion of part of the loan.*

Calculation (XVIII) 9. £57021·21.

(5) *Ascertain the apparent surplus or deficiency in the fund by comparing the value of the present investments representing the fund (1) with the amount found in (4).*

Surplus, £447·27.

(6) *Calculate the corrective instalment, being the annuity which might now be purchased with the amount found in (5), for the unexpired portion of the approximate repayment period (2) at the rate of accumulation (1).*

Calculation (XVIII) 10. £57·45.

(7) *Calculate the annual sinking fund instalment which, if set aside for the unexpired portion of the approximate repayment period (2), would provide the portion of the loan converted into ordinary share capital or stock.*

Calculation (XVIII) 8. £4429·52.

[Here refer to the memo. after (12).]

(8) *Deduct from the annual sinking fund instalment (£7441·63) found as in (3), but not from the fixed instalment (£7,500) originally specified in the trust deed and actually set aside, the annual instalment (£4429·52) found in (7).*

(9) *The remainder (£3012·11), after adjustment in respect of the corrective instalment (£57·45) found in (6) will be the future reduced annual sinking fund instalment, to be set aside and accumulated during the whole of the unexpired portion of the approximate repayment period found in (2).* *£2954·66.*

(10) *Prepare a statement showing the final repayment of the loan by the operation of the fund based (not upon the annual instalment (£7,500) originally fixed), but upon the annual instalment (£7441·63) found in (3), as subsequently reduced by the annual instalment (£4429·52) found in (7) and adjusted by the annual instalment (£57·45) found in (6).* *Statement XVIII. D.*

(11) *If the annual instalment (£7441·63) as found in (3) differs considerably from the annual instalment (£7,500) originally specified in the trust deed, an adjustment may be made which will have the effect of slightly increasing or reducing the annual instalment found in (9), as hereafter described.*

(12) *Prepare the pro forma account mentioned in the previous methods.* *Pro forma Account, No. 6.*

Memo. If the above method be used to adjust a surplus in the fund arising in consequence of the payment into the fund of the proceeds of realisation of part of the assets, as described in Variation II (Surplus) in Chapter XVII, but in which the annual instalment is a stated sum not found by calculation, substitute operation (2) in that method for the above operation (7), namely:—

(2) *Calculate the annuity which may be purchased for the unexpired portion of the repayment period with the amount now paid into the fund.* *Calculation (XVII) 1.*

General Remarks as to the Sinking Funds of Commercial and Financial Undertakings.

In the case of local authorities the method by which loans are required to be repaid by means of a sinking fund is well defined, but in the case of commercial or financial undertakings the conditions are much more variable, and the trust deed may stipulate that it shall be provided either by:—

(1) An equal annual instalment to be calculated on the basis of a given repayment period and a prescribed rate of accumulation, similar to those of local authorities, or

(2) A stated sum to be set aside each year.

Both methods must be considered. In order to attract investors a commercial or financial undertaking, when inviting subscriptions for bonds, debentures, debenture stock, or loan capital of any other nature, may give the investor the option at a future date, which may be specified or not, of converting the loan into share capital or stock of the undertaking, on what may then be very advantageous terms if the concern be making good profits. In the meantime a sinking fund is required by the trust deed to be set aside out of profits in order to repay the total loan on a given date, the fund to accumulate at a rate per cent., which may be specified or not, by investment in outside securities.

During the earlier years, if profits are low, the provision of the annual instalment will have the effect of reducing the dividends which may be paid to the ordinary shareholders, and there will not therefore be any inducement to the loan creditors to give up their security. But a time may come when the position of the undertaking has been materially improved, and if the profits have been good and are likely to continue so, some of the loan creditors may be induced to convert their holding into ordinary share capital or stock. The amount of loan so converted will, of course, correspondingly reduce the amount to be finally provided by means of the sinking fund, and, seeing that the period of repayment of the balance of the loan will remain unchanged, the effect of the partial conversion will be seen solely in a reduction in the future annual instalment to be set aside out of profits during the unexpired portion of the original repayment period. This reduction in the future annual instalment arises in consequence of two factors, namely, the amount of loan withdrawn from the operation of the fund by reason of its conversion into ordinary share capital or stock; and, further, from the fact that the amount now in the fund represents the accumulation of past instalments set aside to provide the whole of the loan. Stated in terms of the balance of loan still unconverted, there is a present surplus in the fund due to setting aside in the past what will in future be excessive annual instalments, and there is also an excessive future annual instalment, both of which factors have been dealt with individually in previous chapters. They are here combined; and the problem is further complicated by the nature of the

annual instalment. If it be a stated sum, it is very probable that it was fixed originally with only an approximate regard to the period of repayment, and it is therefore necessary to ascertain not only the future period of repayment but also the future rate of accumulation. This future rate of accumulation may be based upon the rate of income now yielded by the present investments of the fund, and therefrom it is possible to calculate the period of repayment, both of which are governing factors in the adjustment to be made. The problem may be further complicated by other variations in the period or rate per cent., or by a combination of both, but attention will be directed only to the above factors.

VARIATION III (SURPLUS), arising on the withdrawal of part of the loan from the operation of the sinking fund of a commercial or financial undertaking owing to the conversion of such part of the loan into ordinary share capital or stock of the undertaking:—

in which the original annual instalment was found by calculation, based upon a specified period of repayment and rate of accumulation:

Statement XVIII. A.

The above variation will be illustrated by the information previously obtained with regard to the imaginary sinking fund already discussed, namely, the same amount of original loan, £26,495, repayment period 25 years, rate of accumulation $3\frac{1}{2}$ per cent., amount in the fund £9,463, or a deficiency of £469·74. The assumed conversion of part of the loan takes place at the end of the 12th year and affects £5,000 of the loan. The original annual instalment, £680·234 was arrived at by Calculation (XV) 1.

Statement XVIII, A, following, shows the successive steps in the adjustment of the annual instalment, and Statement XVIII, B, shows the ultimate repayment of the loan by the operation of the fund after making such adjustment.

The above deficiency of £469·74 has been purposely introduced into this example in order to demonstrate that the method adopted will apply to a combination of factors requiring the adjustment. It will illustrate the remark made in a previous chapter that it is not absolutely necessary to ascertain the exact amount of the deficiency at the time of making the adjustment seeing that the calculation is based upon the actual amount now in the fund and the accumulation thereof at the

future rate per cent. This is clearly shown by the examples worked out in this and other chapters by the annual increment (balance of loan) method in which no mention is made, or account taken, of any surplus or deficiency in the amount in the fund as compared with the amount which should be in the fund at the time of making the adjustment.

The reduction of £310·308 in the original annual instalment is the sole effect of the withdrawal of the £5,000 of loan from the operation of the fund, since there is not in this instance any increase in the income to be added to the fund, as found in Chapter XVII was the effect of the payment into the fund of the sum of £4,560 arising out of the proceeds of sale of part of the security for the loan. See Statement XVII. A.

This method of adjusting a surplus in a sinking fund, owing to the withdrawal of part of the loan from the operation of the fund, should therefore be carefully compared with the method found necessary in the case of a surplus arising from a cash payment into the fund from proceeds of assets realised. This will be fully considered at the end of this chapter.

THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. The method of arriving at the amended annual instalment based upon the future annual increment is summarised at the beginning of Chapter XV, and is fully described in Chapter XVI. As the method about to be discussed is based upon the same conditions as in the previous example, Statement XVIII. B., showing the final repayment of the loan will again apply. This method is shown in Statement XVIII. C. following.

A Surplus in the Fund. Statement XVIII. A.

The Deductive Method.

Showing the method of adjusting the annual instalment in consequence of a surplus in the fund, arising on the withdrawal of part of the loan from the operation of the fund owing to the conversion of such part of the loan into ordinary share capital or stock.

Variation III (Surplus), in which the original annual instalment was found by calculation, based upon a specified period of repayment and rate of accumulation.

Calculation (XV) 1.

	Annual instalment.	Equivalent amount of original loan.
Amount of original loan		£26495·00
Original annual instalment. Calculation (XV) 1	£680·234	
Additional annual instalment to provide present deficiency in the fund. Calculation (XV) 3	£45·594	
	£725·828	
Present investments (at end of 12th year) £9463·00		
Amount thereof, accumulated for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 4		£14799·71
	£725·828	£11695·29
Loan withdrawn from the operation of the fund at the end of the 12th year		£5000·00
equivalent to a reduction in the annual instalment of Calculation (XVIII) 1	£310·308	
	£415·520	£6695·29
Amended annual instalment of £415·520 will provide £6695·29 in 13 years at $3\frac{1}{2}$ per cent. Calculation (XVIII) 2	£415·520	£6695·29

A Surplus in the Fund. Statement XVIII. B.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the sinking fund after making the adjustment in the annual instalment consequent upon a surplus in the fund, arising on the withdrawal of part of the loan from the operation of the fund owing to the conversion of such part of the loan into ordinary share capital or stock.

VARIATION III (SURPLUS), in which the original annual instalment was found by calculation, based upon a specified period of repayment and rate of accumulation.

Calculation (XV) 1.

		Annual instalment	Equivalent amount of original loan.
Present investments (at end of 12th year)	£9463·00		
Amount thereof, accumulated for 13 years at $3\frac{1}{2}$ per cent. Calculation (XV) 4			£14799·71
Amended annual instalment:—			
Original annual instalment		£680·234	
Additional annual instalment to provide the present deficiency of £469·744. Calculation (XVI) 1		£45·594	
		£725·828	
Reduced annual instalment due to withdrawal of £5,000 of loan. Calculation (XVIII) 1		£310·308	
		£415·520	
Amount thereof, in 13 years at $3\frac{1}{2}$ per cent. Calculation (XVIII) 2			£6695·29
Amount in the fund at the end of 25 years			£21495·00
being amount of original loan ...		£26495·00	
less the amount converted as above		£5000·00	
			£21495·00

A Surplus in the Fund. Statement XVIII. C.

The Annual Increment (balance of loan) Method.

To find the amended annual sinking fund instalment consequent upon a surplus in the fund, arising on the withdrawal of part of the loan from the operation of the fund, owing to the conversion of such part of the loan into ordinary share capital or stock.

VARIATION III (SURPLUS), in which the original annual instalment was found by calculation, based upon a specified period of repayment and rate of accumulation.

Calculation (XV) 1.

Amount of original loan (25 years)		£26495·00
deduct portion thereof converted into ordinary share capital or stock and withdrawn from the operation of the fund at the end of the 12th year		£5000·00
		£21495·00
deduct amount in the fund at the end of the 12th year		£9463·00
Balance of loan		£12032·00
Amended annual increment to be added to the fund, and accumulated at 3½ per cent. to provide this amount at the end of 13 years Calculation (XVIII) 3		£746·725
deduct income to be received from the present investments (£9,463) at 3½ per cent.		£331·205
Amended annual instalment, *being*		£415·520
Original annual instalment	£680·234	
increased by	£45·594	
	£725·828	
and reduced by	£310·308	
		£415·520

The rule relating to this method is stated at the head of Chapter XXII.

Pro forma Sinking Fund Account, No. 5.

A Surplus in the Fund. (Variation III.)

Loan of £26,495, repayable at the end of 25 years.

Showing the final repayment of the loan, by the operation of the reduced annual instalment of £415·520.

Statement XVIII. B. Rate of accumulation, $3\frac{1}{2}$ per cent.

Year.	Amount in the fund at beginning of year.	Income received from investments $3\frac{1}{2}$ per cent	Annual sinking fund instalment.	Amount in the fund at end of year.	Year.
1					1
2					2
3					3
4	The amount in the fund at the end of the 12th year, £9,463, is an assumed amount, and is equivalent to setting aside an annual instalment of £648·064, as shown by Calculation (XVI) 10, instead of the correct annual instalment of £680·234.				4
5					5
6					6
7					7
8					8
9					9
10					10
11					11
12				9463·000	12
13	9463·000	331·205	415·520	10209·725	13
14	10209·725	357·340	415·520	10982·585	14
15	10982·585	384·390	415·520	11782·495	15
16	11782·495	412·387	415·520	12610·402	16
17	12610·402	441·364	415·520	13467·286	17
18	13467·286	471·355	415·520	14354·161	18
19	14354·161	502·396	415·520	15272·077	19
20	15272·077	534·523	415·520	16222·120	20
21	16222·120	567·774	415·520	17205·414	21
22	17205·414	602·189	415·520	18223·123	22
23	18223·123	637·809	415·520	19276·452	23
24	19276·452	674·676	415·520	20366·648	24
25	20366·648	712·832	415·520	21495·000	25

VARIATION IV. (SURPLUS), arising on the withdrawal of part of the loan from the operation of the sinking fund of a commercial or financial undertaking owing to the conversion of such part of the loan into ordinary share capital or stock of the undertaking:—

in which the original annual instalment is a stated sum and is not based except in a general way upon any period of repayment or rate of accumulation.

Statement XVIII. D.

This variation is similar in principle to the one last discussed, but requires different treatment owing to the fact that the original annual instalment is a stated sum arrived at in a somewhat empirical manner, without any calculation similar to (XV) 1, which is based upon a prescribed period of repayment and rate of accumulation. In the following example, used to illustrate the variation under review new data have been adopted, and the question is dealt with solely as regards the loan debt of commercial or financial undertakings without making any comparison with the methods to be adopted in the case of a local authority. In the early days of municipal finance the annual instalment to be set aside was often a fixed amount, being generally a definite percentage of the amount of the loan, but the conditions then imposed upon such authorities were vague and indefinite both as regards the accumulation of the fund and the gradual repayment of the debt, and left entirely out of account the life or duration of utility of the asset created out of the loan.

EXAMPLE TO ILLUSTRATE VARIATION IV (SURPLUS). The sinking fund under review relates to the repayment of a loan of £150,000, and an annual instalment of £7,500 is required to be set aside for this purpose out of the profits of the undertaking and invested in outside securities. Under the trust deed the loan creditors have the option of converting their holding into ordinary share capital or stock at any time within seven years from the date of issue. The price payable, on conversion, for the ordinary share capital is immaterial for the present purpose, as is also the rate of interest payable upon the loan; but any premium payable to the loan-holders upon conversion or redemption should be taken into account. At the end of the seventh year the holders of £45,000 of loan elect to exercise the above option, and convert their loan holding into ordinary share capital or stock. Seeing that no specified period is

prescribed within which the loan shall be repaid by means of the sinking fund and that the annual sum to be set aside is fixed at £7,500, there was not any necessity, at the date of issue of the loan, to make any calculation of the annual instalment as in the case of the sinking funds of local authorities. This annual instalment of £7,500, it will be assumed, has been regularly set aside and invested each year, and at the end of the seventh year, when £45,000 of original loan is converted into ordinary share capital or stock, it will have amounted to £57468·48, having earned an average accumulation rate of 3 per cent., as shown by Calculation (XVIII) 4. In actual practice, of course, this amount would be obtained from the actual records or books of account.

The position at the end of the seventh year will therefore be as follows:—

1. Loan outstanding and unconverted	£105,000
2. Amount in the sinking fund, invested and yielding 3 per cent. per annum	£57468·48
3. Present annual instalment	£7500·

This annual instalment will be reduced in future years owing to the withdrawal of £45,000 of loan from the operation of the fund.

The next step in the adjustment is to ascertain the annual amount by which this instalment may be reduced and yet fulfil the original obligation to repay the unconverted portion of the loan under the original conditions. There are several ways of doing this, as may be gathered from previous examples. But it is in any case first essential that the estimated future rate of accumulation shall be fixed. In this case past experience is available, and, for convenience, 3 per cent. will be taken, being the rate of income already yielded by the present investments representing the fund. Any variation in this rate per cent. and in the future accumulation rate may be treated as explained in Chapter XXI (variation in the rates per cent.). Having decided upon the future estimated rate of accumulation, it is next necessary to fix the period of redemption in order to ascertain the reduction in the annual instalment of £7,500. This would not be necessary but for the amount at present in the fund. The instalment then would be $^{105}/_{150}$ths of the original instalment of £7,500, or £5,250 per annum, but this

will be reduced by an annual amount depending upon the money now in the fund. There are therefore two factors to be taken into account—(1) the accumulation of the £57468·48 now in the fund, and (2) the accumulation of the future reduced annual instalment which it is required to ascertain. It is not possible to combine, in one calculation, factors involving the amount of £1, and also of £1 per annum without reducing both to a common denomination, and therefore it is better as the simplest method to revert to the original conditions at the date of the issue of the loan.

The first step is to ascertain the approximate number of years in which an annual instalment of £7,500 will amount to, and repay, a loan of, £150,000 if accumulated at 3 per cent. per annum. This is the rate of income which has been yielded by the present investments representing the fund and which it is assumed will continue to be yielded by any future investments. The number of years may be ascertained approximately by an inspection of Table III, seeing that if £7,500 per annum will, at 3 per cent., amount to £150,000, £1 per annum will, at the same rate and in the same time, amount to £20. Table III gives the following figures:—

£1 per annum will, at 3 per cent., amount to—

in 15 years	£18·59891,
in 16 years	20·15688,
in 17 years	21·76159,

and an even 16 years is therefore adopted as the approximate period of repayment, which will be slightly in excess of the actual period required, and the calculated annual instalment will therefore be less than £7,500. In order to make the calculation in such a manner that the result may be proved as in other cases, it is necessary to first ascertain the exact annual instalment, to be accumulated at 3 per cent., to repay £150,000 in exactly 16 years. The annual instalment is £7441·63, as shown by Calculation (XVIII) 5.

This annual instalment of £7441·63 is less than the stated annual instalment of £7,500, as will be gathered from the above extracts from Table III, which show that £1 per annum will in 16 years, at 3 per cent., amount to £20·15688; consequently the prescribed annual instalment of £7,500 will amount to:—

$$(£20·15688 \times £7,500) \text{ or to } £151,176·59$$

in 16 years at 3 per cent., as shown by Calculation (XVIII) 6.

Expressed in terms of the above difference it will be seen that:

£7,500 − £7441·63 per annum or £58·37

will in 16 years, at 3 per cent., amount to

£151,176·59 − £150,000 or £1176·58

as shown by Calculation (XVIII) 7.

By adopting the above annual instalment of £7441·63 instead of £7,500, an intentional error of £58·37 per annum is introduced, relating to the repayment of a loan of £150,000 in 16 years. But the reduced annual instalment which is required, and which will be based upon the above annual instalment of £7441·63, will relate to a loan of £105,000 repayable in 9 years only. This intentional error may be corrected if thought desirable or required in the manner to be afterwards explained.

The following data have now been ascertained:— A loan of £105,000 is repayable in a period of 9 years, and towards this there is in the fund an amount of £57468·48, which, it is estimated, will accumulate at 3 per cent. There is an annual instalment of £7441·63 to be set aside for 9 years, and an intentional error of £58·37 per annum has been introduced into the problem.

It is required to find the annual amount by which the above instalment of £7441·63 may be reduced, consequent upon the withdrawal of £45,000 of loan from the operation of the fund.

The problem differs somewhat from the surplus of £4,560, already considered in Chapter XVII (Statement XVII. A.). In that case the £4,560 was paid into the fund in consequence of the realisation of assets forming part of the security for the loan, and was applied in repaying part of the loan or remained to swell the assets of the fund.

In the present instance the conversion of £45,000 of loan into share capital or stock may be looked upon as an entirely separate transaction, and may be regarded as so much cash received in consequence of the issue of new share capital, and applied in reduction of the loan debt. The undertaking, except as afterwards mentioned, does not derive any benefit from the substitution of its obligation to the new shareholders for its obligation to the previous loanholders. Indeed, it may happen that the inducement to the loanholders to convert their secured debt into ordinary share capital or stock is the expectation of a higher rate of interest upon their investment. The only benefit to the undertaking is that the capital is firmly invested in the concern; and the saving in the sinking fund instalment, if previously taken out of profits, will help to provide any increased return payable by way of dividend to the

original loanholders in respect of the loan capital so converted. It is here necessary to depart from the method adopted in dealing with the sum of £4,560 paid into the fund in the example considered in Chapter XVII. The £4,560 was an amount actually in hand, and Calculation (XVII) 1 shows the method of finding the future annuity which it would purchase. The £45,000 in the present case, on the contrary, is an amount due at a future time, namely, at the end of the sinking fund period, and the problem therefore becomes inverted, and instead of calculating the annuity which £45,000 will purchase, it is required to ascertain the annuity or annual sinking fund instalment which in the remaining unexpired period of 9 years will amount to that sum. This is shown by Calculation (XVIII) 8, which may be usefully compared with Calculation (XVII) 1. The annual instalment so found is £4429·52, by which amount the original annual instalment of £7441·63 may be reduced, making the amended annual instalment £3012·11.

There is, however, a further slight correction to be made. In order to simplify the calculation an even period of 16 years has been adopted, which is in excess of the actual period of repayment and requires a reduced annual instalment of £7441·63 instead of £7,500. Until the conversion of part of the loan into ordinary share capital or stock, the undertaking had been setting aside an annual instalment of £7,500, so that there is now an apparent surplus in the amount in the fund as compared with what would have been in the fund if £7441·63 only had been annually set aside. To ascertain the amount of this surplus it is requisite to ascertain the amount to which an annual instalment of £7441·63 will accumulate in 7 years at 3 per cent.

This, as shown by Calculation (XVIII) 9, is	£57021·21
and on comparing this sum with the amount actually in the fund, being the accumulation of the stated instalment of £7,500 Calculation (XVIII) 4, viz.:	£57468·48
the apparent present surplus is found to be	£447·27

which amount, being now in the fund, will accumulate for 9 years at 3 per cent., and is the present value of an annuity, as shown by Calculation (XVIII) 10, of £57·4446 which may be applied in further reduction of the annual instalment of £7441·63 in the same way that the annual instalment of £442·601 was applied in the case of the surplus of £4,560 in Chapter XVII, Statement XVII, A.

In the event of the calculated instalment found as above exceeding the prescribed instalment of £7,500 there would be an apparent deficiency in the fund, instead of a surplus, which would alter the method, but not the principle, of the minor adjustment under consideration.

The various stages of the adjustment have already been so fully described that it is not requisite to prepare a statement similar to XVIII, A, in the previous example.

A statement has, however, been prepared, similar to XVIII, B, showing the final repayment of the loan by the operation of the fund after making the above adjustment in the annual instalment. See Statement XVIII, D, and the pro forma account No. 6 following.

Correction of the Intentional Error. There now only remains the correction of the above intentional error of £58·37 in taking the annual instalment at the calculated amount of £7441·63 instead of £7,500, which lengthened the period of repayment by part of a year.

The £7,500, or any other similarly prescribed annual instalment, is generally fixed in an empirical manner with only a rough approximation to the actual requirements based upon the conditions in each case. It may therefore be concluded that the instalment ascertained in the above manner will meet any practical need likely to arise in such a case. If, however, there is at any time a necessity for greater accuracy it may be ascertained, approximately, by the following method:—

The intentional annual error introduced was	£58·37
This caused an apparent surplus in 7 years of	£447·27
Equal to an annual instalment spread over 9 years of	£57·45
This error related to a loan of	£150,000
The correction will relate to a loan of only	£105,000
Therefore the correction may be taken as $^{105}/_{150}$ths of £57·45, or	£40·215
which would in 9 years amount, at 3 per cent., to Calculation (XVIII) 13	£408·549
The present value of this annual sum in 9 years, at 3 per cent., is Calculation (XVIII) 14	£313·118

and the correction may be made by increasing each of the reduced annual instalments of £2954·66 by £40·215, or by paying into the fund at the present time the above present value thereof, namely, £313·118. If no such correction be made, only £408·549 of original loan will remain unprovided for at the end of the period (which is slightly less than 16 years) in which the original instalment of £7,500 would have repaid the loan.

The Annual Increment (balance of loan) Method. The method of arriving at the amended annual instalment based upon the future annual increment is summarised at the beginning of Chapter XV, and is fully described in Chapter XVI. As the method about to be discussed is based upon the same data as in the example previously used, the following Statement XVIII, D, showing the final repayment of the loan will still apply. For the reasons already given the calculation cannot be made in terms of the stated instalment, but must be made in terms of the approximate amount of £7441·63 found by Calculation (XVIII) 5. This method is shown in Statement XVIII, E, following.

A Surplus in the Fund. Statement XVIII. D.

Showing the final repayment of the loan, by the operation of the sinking fund, after making the adjustment in the annual instalment, consequent upon a surplus in the fund, arising on the withdrawal of part of the loan from the operation of the fund owing to the conversion of such part of the loan into ordinary share capital or stock.

Variation IV (Surplus), in which the original annual instalment is a stated sum, and is not based, except in a general way, upon any period of repayment or rate of accumulation.

	Annual instalment.	Equivalent amount of original loan.
Present investments (at end of 7 years), Calculation (XVIII) 4	£57468·48	
Amount thereof, accumulated for 9 years at 3 per cent. Calculation (XVIII) 11		£74983·30

Amended annual instalment:—

Original annual instalment, as provided by trust deed	£7500·00	
Substituted annual instalment as adopted in Calculation (XVIII) 5, based upon a rate of accumulation of 3 per cent. and a repayment period of 16 years	£7441·63	
This will be reduced by the annual instalment required to repay the £45,000 of loan withdrawn, in 9 years at 3 per cent. Calculation (XVIII) 8	£4429·52	
	£3012·11	
And will be further reduced by the annual instalment to provide £447·27, being the surplus which will be in the fund at the end of 16 years, due to taking an even period of 16 years Calculation (XVIII) 10	£57·45	
	£2954·66	
Amount thereof, accumulated for 9 years at 3 per cent. Calculation (XVIII) 12		£30016·70
Amount in the fund, at the end of 16 years		£105000·00
being the original loan	£150000·00	
reduced by the amount of loan withdrawn from the operation of the fund ...	£45000·00	
		£105000·00

This statement shows the method of making the correction in the annual instalment, and also the final repayment of the loan. The amended annual instalment may also be found by the annual increment (balance of loan) method, as shown in Statement XVIII. E.

A Surplus in the Fund. Statement XVIII. E.

The Annual Increment (balance of loan) Method.

To find the amended annual sinking fund instalment, consequent upon a surplus in the fund, arising on the withdrawal of part of the loan from the operation of the fund owing to the conversion of such part of the loan into ordinary share capital or stock.

VARIATION IV (SURPLUS), in which the original annual instalment is a stated sum, and is not based, except in a general way, upon any period of repayment or rate of accumulation.

Amount of original loan (16 years)		£150000·00
deduct portion thereof converted into ordinary share capital and withdrawn from the operation of the fund at the end of the 7th year		£45000·00
		£105000·00
deduct amount in the fund at the end of the 7th year		£57468·48
Balance of loan		£47531·52
Amended annual increment to be added to the fund, and accumulated at 3 per cent. to provide this amount at the end of 9 years Calculation (XVIII) 15		£4678·71
deduct income to be received from the present investments (£57468·48) at 3 per cent.		£1724·05
Amended annual instalment, *being* :—		£2954·66
Calculated instalment	£7441·63	
reduced by	£4486·97	
		£2954·66

The final repayment of the loan by the operation of the sinking fund, after making the above adjustment in the annual instalment is shown in Statement XVIII. D.

Pro forma Sinking Fund Account, No. 6.

A Surplus in the Fund. (Variation IV.)

Loan of £150,000, repayable by a stated annual instalment of £7,500.

SHOWING THE FINAL REPAYMENT of the balance of unconverted loan, by the operation of the reduced annual instalment of £2954·660.

Statement XVIII. D. Rate of accumulation. 3 per cent.

Year.	Amount in the fund at beginning of year.	Income received from investments 3 per cent.	Annual sinking fund instalment.	Amount in the fund at end of year.	Year.
1	Nil	Nil	7500·000	7500·000	1
2	7500·000	225·000	7500·000	15225·000	2
3	15525·000	456·750	7500·000	23181·750	3
4	23181·750	695·453	7500·000	31377·203	4
5	31377·203	941·316	7500·000	39818·519	5
6	39818·519	1194·556	7500·000	48513·075	6
7	48513·075	1455·495	7500·000	57468·480	7
8	57468·480	1724·054	2954·660	62147·194	8
9	62147·194	1864·416	2954·660	66966·270	9
10	66966·270	2008·988	2954·660	71929·918	10
11	71929·918	2157·898	2954·660	77042·476	11
12	77042·476	2311·274	2954·660	82308·410	12
13	82308·410	2469·252	2954·660	87732·322	13
14	87732·322	2631·970	2954·660	93318·952	14
15	93318·952	2799·569	2954·660	99073·181	15
16	99073·181	2972·159	2954·660	105000·000	16
			Amount of loan converted	45000·000	

COMPARISON OF METHODS PREVIOUSLY DISCUSSED. This concludes the examination of the various methods of adjusting a sinking fund to compensate for a difference between the actual amount in the fund and the amount which should be in the fund at any time in order to carry out the original obligation; and the results may be summarised as follows.

In the case of the adjustment (Variation I) caused by a deficiency in the fund, as shown by Statement XVI, A, in Chapter XVI, the deficiency was corrected by an additional sinking fund instalment to be set aside during the whole of the unexpired portion of the repayment period, as shown by Calculation (XVI) 1.

In Variation II, described in Chapter XVII, Statement XVII, A, relating to a surplus of £4,560, being the proceeds of sale of assets, paid into the fund, a different method was adopted. In that case there was an actual increase in the cash assets of the fund which operated in two ways, (1) by increasing the future income of the fund, in consequence of which the present sum of £4,560 will ultimately repay, by accumulation, £7131·64 of original loan, and (2) by reducing the amount of loan ultimately repayable by £7131·64, it relieved the future years of the amount of the sinking fund instalment (£442·60) equivalent to that amount for the unexpired portion of the repayment period of 13 years, which is the annuity which might now be purchased with the sum of £4,560. In the two Variations III and IV which have just been considered there is a surplus in the fund caused by the withdrawal of part of the loan from the operation of the fund. There is here no actual addition to the assets of the fund, as in the case of the payment into the fund of the proceeds of sale of part of the security for the loan. The surplus may in effect be considered as a lightening of the burden previously borne by the undertaking measured by the reduction in the amount of loan to be ultimately provided. Consequently the surplus operates in one direction only, namely, by reducing the original annual instalment to be set aside, whether that instalment was arrived at by calculation in the ordinary manner or was a round sum specified in the trust or other deed under which the fund was instituted.

Further Problems. There are other problems which may arise in connection with the sinking funds of commercial or financial undertakings, but which have not been treated in an exhaustive manner, because they may be solved by one or other of the methods elsewhere described. They are as follows:—

Redemption by Drawings:

If annual, they may be considered on the lines of the instalment method of local authorities. (Chapter XI.)

If at periods of years, a sinking fund may be set aside during each period to provide the proportion of the loan repayable at the end of each period.

If at periods of years, in a series, a sinking fund may be provided by setting aside and accumulating equal annual amounts during the whole period in order to provide the amounts repayable at the end of each period. This will apply to the simultaneous provision out of profits of loans repayable in certain priorities.

REDEMPTION OF LOANS (ISSUED AS STOCK) AT A PREMIUM:

If the premium be stated, the sinking fund instalment should be calculated to provide that amount in addition to the par value, and there is not any change in the method described.

If the premium depends upon the price at the date of redemption, and cannot be accurately estimated, the annual instalment should be based upon the par value of the stock, and the premium provided for, as and when it arises, by charging it to revenue account, or by making prudent provision in anticipation.

REDEMPTION OF LOAN IN PART.

The trust deed may provide that if any part of the loan be redeemed out of the fund, the interest previously paid upon such redeemed loan shall be added to the fund, although the rate of interest payable to the loanholder be higher than the calculated rate of accumulation of the fund. This will cause a surplus in the fund over the calculated amount, which will have the effect of anticipating the final maturity of the fund, whether the loan is repayable on a specified date or by the accumulation of a stated instalment. The possibility of making any provision for such an event when calculating the original instalment in the case of an ordinary sinking fund will depend upon the circumstances of each individual case.

CESSATION OF ANNUAL CONTRIBUTIONS. Instead of making the adjustment by spreading any surplus, however arising, equally over the unexpired portion of the repayment period, it may be provided that the amount in the fund shall continue to accumulate, and the original instalments be annually paid in, until such time as the fund is of such an amount that the

present investments and the accumulations of the annual income to be received therefrom in future will be sufficient, without any further instalments, to provide the amount of loan repayable. (See Article 11 (2) County Stock Regulations, 1891.)

CONTINUATION OF INSTALMENTS. It may be provided that the original instalment shall continue to be set aside and added to the fund until the loan is ultimately repaid, notwithstanding :

(1) The withdrawal of any part of the loan from the operation of the sinking fund by reason of its being converted into ordinary share capital or stock.

(2) The sale of any part of the assets forming part of the security for the loan, and the payment of the proceeds into the fund.

(3) Any other cause operating to produce a surplus in the fund or to accelerate the date of maturity of the fund.

In such cases it may be necessary to determine the reduced period of redemption which may be ascertained by one or other of the methods described.

CHAPTER XIX.

SINKING FUND PROBLEMS, RELATING TO THE RATE PER CENT.,

OF INCOME UPON THE PRESENT INVESTMENTS REPRESENTING THE AMOUNT IN THE FUND; AND ALSO THE FUTURE RATE OF ACCUMULATION OF THE FUND.

VARIATION A, IN WHICH THERE IS A VARIATION IN THE RATE OF ACCUMULATION WITHOUT ANY VARIATION IN THE RATE OF INCOME UPON THE PRESENT INVESTMENTS, OR IN THE PERIOD OF REPAYMENT. STATEMENT XIX. B.

SUMMARY OF THE METHODS OF ADJUSTMENT. GENERAL CONSIDERATIONS AS TO VARIATIONS IN THE RATE PER CENT. TO BE TREATED IN DETAIL IN THE FOLLOWING CHAPTERS. THE DEDUCTIVE METHOD. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN BY THE OPERATION OF THE AMENDED ANNUAL INSTALMENT.

Summary of the methods of adjustment.

(I) The deductive method, as summarised below, is of wider application than the variation in the rate of accumulation only, and has been so worded that it may be treated as the standard method relating to all variations. Statement XIX. A.

(II) The direct method, without calculation, as summarised at the head of Chapter XX, will not apply to this variation.

(III) The annual increment (balance of loan) method, as summarised at the head of Chapter XXII, may be used, but will not be applied to the example under review. The method of finding the amended annual increment is shown in Calculation (XIX) 5.

(IV) The annual increment (ratio) method, as summarised at the head of Chapter XXIII. Statement XXII. C.

Note. The terms used in the following summary are fully explained at the head of Chapter XXII. In all the above methods, it is imperative that the rate of accumulation and of income from investments be uniform during the whole of the unexpired or substituted portion of the repayment period.

SUMMARY OF THE DEDUCTIVE METHOD, *of ascertaining the amended annual sinking fund instalment due to a variation in the rate per cent. of accumulation, accompanied by, or without, any variation in the rate of income to be received upon the present investments representing the fund, and also due to any variation in the period of repayment, or any combination of the above factors.* *Statement XIX. A.*

(1) *Ascertain the value of the present investments in the manner already described, and also the amount of the present annual income yielded by such investments, up to the time of making the adjustment.*

(2) *To the present annual income, so ascertained, add the present or original annual instalment which has been set aside and added to the sinking fund up to the time of making the adjustment.*

(3) *The total so obtained is the present annual increment of the fund.*

(4) *Ascertain, or estimate, the rate per cent. at which the fund will accumulate in future (the future rate).*

(5) *Calculate (in one sum or separately) the amount of the present annual increment found, as in (3), for the number of years in the unexpired or substituted period of repayment, at the future rate of accumulation fixed in (4).* *Calculations (XIX) 1 and 2.*

(6) *The amount or amounts, so ascertained, will represent the portion of original loan which will be provided at the end of the original or varied period of repayment.*

(7) *To this amount add the value of the present investments, as ascertained in (1), and deduct the sum from the amount of the original loan.*

(8) *The remainder represents the portion of original loan which is now unprovided for by the present investments and the future accumulation of the present annual increment found in (3).*

(9) *Calculate the additional annual sinking fund instalment which, at the future rate of accumulation, estimated as in (4), will amount to the balance of loan found in (8) at the end of the unexpired or substituted period of repayment.* *Calculation (XIX) 3.*

(10) *This additional annual instalment, added to the present annual increment found in (3) gives the same future or amended annual increment, which is found by direct calculation by the annual increment (ratio) method.*

(11) *From the future, or amended annual increment, so ascertained, deduct the future annual income from the present investments; and the remainder is the future or amended annual instalment to be charged to revenue or rate in substitution for the present or original annual instalment.*

(12) *Prepare a statement showing the final repayment of the loan by the operation of the fund under the amended conditions.* *Statement XIX. B.*

(13) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired or substituted period of repayment.*
Pro forma Account, No. 7.

Memo. The above method is worded to apply to a reduction in the rate of accumulation or other factor, but it will apply equally to an increase in such factors with very little modification. It should be compared with the deductive method summarised at the head of Chapter XXIV.

GENERAL CONSIDERATIONS AS TO THE RATE PER CENT.

Having described the various methods of dealing with problems arising out of a deficiency or a surplus in the sinking fund, further questions will now be considered in connection with the rate per cent., beginning with cases in which it is anticipated that the original estimated rate of accumulation will not be realised in future. This is mainly due to a fluctuation in the money market of a more or less permanent character affecting the future return on all investments. Questions will also arise in consequence of a reduction in the rate of income to be received in future on investments already made, as was the case in 1888, when, under Mr. Goschen's Finance Act, the rate of Consols was reduced from 3 per cent. to $2\frac{3}{4}$ per cent. for 15 years, after which a further reduction to $2\frac{1}{2}$ per cent. took place. Other causes may operate in a similar manner, especially in the case of commercial and financial undertakings.

The problem will differ according as the variation in the original conditions affects:—

(1) The rate of accumulation anticipated to be realised on the investment of future accretions to the fund. *Variation A.*

(2) The rate of income to be received on the present investments representing the fund. *Variation B.*

(3) Both the above rates in combination. *Variation C.*

In making the adjustments it will at times be difficult to forecast accurately the future rate of income to be received on the present investments. In such cases it is wise to form a conservative estimate of the future rate and fix it on the low side; or to take a slightly lower rate of accumulation and thereby increase the annual instalment to be charged to revenue or rate account. In discussing the following variations it will be assumed that although the future rate of income to be received upon the present investments will change, yet it will be uniform during the whole of the unexpired repayment period. But cases may arise in which this will not be so, but in which the rate of income will again vary, during the term, in a definite manner laid down in advance, as in the case of Consols previously referred to. A variation of this nature, occurring during the unexpired portion of the repayment period, will be deferred to Chapter XXVII. When considering Variation B (rate of income only) in Chapter XX, it will be found that the future rate of accumulation is the most important factor in the adjustment, although it may not be the greater as regards the actual amount of money involved.

The following discussion will be confined to a reduction only in both the above rates per cent., but it should be borne in mind that the method to be adopted and described will apply equally to an increase in both rates or to an increase in one and a decrease in the other. This will be better appreciated after considering the methods of making the adjustment by the annual increment (ratio) method.

Any deficiency in the fund at the time of making the enquiry, and arising out of a reduction in the rate of income received from investments previously made, or from other causes, will not affect the present method of calculation. Any such deficiency may or may not be discovered on ascertaining the present position of the fund as described in the previous chapter. The following method differs from the one there described, in that, in the present example, the basis of the adjustment is the value of the present investments, and not the amount to which they will accumulate at the end of the term.

In dealing with a deficiency, it was assumed that there would not be any variation in the rate of accumulation, whereas in the present example the reduction in the rate of accumulation is the cause of the rectification under discussion.

In an actual enquiry of this nature, the amount in the fund at the end of the 12th year, as shown by the records, would most probably be compared with the calculated amount which should be in the fund according to the pro forma account, and the deficiency or surplus thereby ascertained, but it is not absolutely necessary to do this. The important factors are, the value of the present investments, the future income they may be expected to produce, and the rate of accumulation which will be yielded by the investment of the future accretions to the fund. In this connection Chapter XIV, dealing generally with the present investments and the annual increment should be consulted, especially as to the meaning of the term "present investments." The deductive method will apply to the rectification of a present deficiency or surplus combined with a variation in the future rates of income or accumulation, because in this case the enquiry is based upon the value of the investments now representing the fund; and the method of approaching the problem is not altered because that value is greater or less than the amount which should be in the fund according to the original calculation, and as shown by the pro forma account. The method about to be described will show the amended annual instalment to be charged to revenue or rate, based upon the present state of the fund, but if it be required to allocate this as between a present deficiency or surplus and the future reduction in the rates of income or accumulation, it will be necessary to make, first, the calculation as to the deficiency or surplus, as already described, followed by the enquiry as to the increased annual instalment due solely to the fall in the rate or rates per cent.

Details of the Sinking Fund. The sinking fund which will be used to illustrate all problems relating to a variation in the rate per cent. will apply to a loan of £26,495, repayable at the end of a period of 25 years, requiring an annual instalment of £680·234 to be set aside and accumulated at $3\frac{1}{2}$ per cent. [Calculation (XV) 1], and it will in all cases be assumed, as when considering the rectification of a surplus, that at the end of the 12th year the fund stands at the proper calculated amount of £9932·74, as found by Calculation (XV) 2. This sum is represented by investments worth that amount, which

have up to the present yielded an annual income at the rate of $3\frac{1}{2}$ per cent. per annum, being the original estimated rate of accumulation upon which the above instalment was based. This sum of £9932·74, if accumulated at the above rate of $3\frac{1}{2}$ per cent., will provide for the repayment of £15534·38 of original loan at the end of 25 years, as found by Calculation (XVII) 2.

VARIATIONS IN THE RATE PER CENT., TO BE CONSIDERED IN DETAIL. In order to illustrate the problems to be discussed in this and following chapters three variations from the original conditions as regards the rate of accumulation of $3\frac{1}{2}$ per cent. will be considered. In fixing this rate per cent. in the first instance it was assumed that it would continue to be received upon the whole of the accumulations of the fund during the whole of the repayment period of 25 years. If this anticipation had been realised the rate per cent. of income upon investments and the rate per cent. of accumulation would have been the same in all cases, namely, $3\frac{1}{2}$ per cent., and the fund would have pursued its calculated course until maturity.

In the three examples about to be considered a gradual decrease in the rate of income from investments, as well as in the rate of accumulation, will be assumed to occur between each set of conditions; but when comparing the several results in a later chapter they will be considered only as regards the alteration in the rate of accumulation as follows:—

Chapter.	Variation.	Compared with	Future rate of income on present investments.	Future rate of accumulation.
XIX	A	Original conditions	unaltered	reduced
XX	B	Variation A	reduced	unaltered
XXI	C	Variation A	reduced	reduced

The paramount importance of the rate of accumulation in such problems has already been referred to, and it will be noticed from the above table that Variations (A) and (C) alone contain any variation in that rate. The following details as to each variation are given for convenience of reference and comparison:—

CHAPTER XIX. *Variation (A) in the rate of accumulation only.*
Compared with the conditions at the time the original calculation was made.
In this example the rate of accumulation is reduced from $3\frac{1}{2}$ to 3 per cent., but the rate of income upon the present investments remains at $3\frac{1}{2}$ per cent.

CHAPTER XX. *Variation (B) in the rate of income upon the present investments only.*

Compared with the conditions in Variation (A).

In this example the rate of accumulation is unaltered, and remains at 3 per cent., but the rate of income upon the present investments is reduced from $3\frac{1}{2}$ to 3 per cent.

CHAPTER XXI. *Variation (C) in the rate of accumulation, as well as in the rate of income upon the present investments.*

Compared with the conditions in Variation (A).

In this example the rate of accumulation is reduced from 3 to $2\frac{1}{2}$ per cent., and the rate of income upon the present investments is reduced from $3\frac{1}{2}$ to 3 per cent.

These variations will now be examined, and will be treated as independent problems instead of variations of the same fund. This procedure involves a certain amount of repetition, but is adopted in order to emphasize the principles involved, with the view of finding a shorter method of making the adjustments. There is also a further advantage, namely, that each problem may be studied separately so that any cases occurring in actual practice may be referred to a similar example completely worked out in detail.

It will be noticed on referring to the above details and to the summary of results given in Chapter XXI, Statement XXI, C, that the above variations are not isolated cases without any connection. They are intimately related by design, and form a series commencing with the original conditions and leading by successive stages to Variation C (rate of income and accumulation). When considering the derivation of a rule and formula relating to the adjustment of a sinking fund in consequence of a simultaneous variation in the rates per cent. of accumulation and income on investments these variations will be combined, and in one instance (Calculation XXII, C), Variation A will be inverted to serve as an example of an increase in the rate per cent. of accumulation.

Any decrease in the rate of income yielded by the present investments or by the future investments of the annual accretions to the fund will have the effect of reducing the sum to which the fund will amount at the end of the repayment period. The amount of such deficiency will depend upon the actual rates to be received in future as compared with the

original rate of accumulation, namely, $3\frac{1}{2}$ per cent. It is necessary so to adjust the sinking fund that the deficiency due to a fall in the rate either of income or of accumulation shall not only be made good, but be spread equally over the remaining 13 years, by increasing the original sinking fund instalment by such an annual amount as will be sufficient for the purpose.

THE DEDUCTIVE METHOD. In order to ascertain the amount by which the annual instalment should be increased, the present sinking fund factors may be reduced, either to terms of present value or to equivalent amounts of original loan repayable at the end of the 25 years, but, as in the former example, it is preferable to deal with the figures representing equivalent amounts of loan.

In each of the above variations the common factors are:—

(1) A sum of £9932·74 standing to the credit of the fund at the end of the 12th year, which is invested and expected to realise that sum at the end of the repayment period.

(2) The income arising from the above present investments.

(3) The original annual instalment of £680·234 to be set aside for the unexpired term of 13 years, and which will also be invested each year.

(4) The income to be received annually from (2) and (3) when invested.

Items (2) and (3) constitute the present annual increment of the fund, as described in Chapter XIV and in Chapter XXII.

In each case the original annual instalment of £680·234 will be supplemented by an additional annual instalment to be ascertained, and which, added to the present annual increment, will give the future or amended annual increment of the fund. The method of approaching the solution of the problem is the same in each variation.

A statement will be prepared similar to XIX, A, showing the position of the fund at the end of the 12th year, when the assumed necessity arises to make the adjustment due to a change in the rate per cent. either of income or accumulation or both. This statement will commence with the amount now in the fund, which will be included at its present value without accumulation. This is equivalent to deducting that amount from the original loan, leaving the balance to be provided by the accumulation of the future or amended annual increment which is composed of the future income from the present investments and the amended annual instalment.

This is a departure from the procedure followed previously in dealing with a surplus or a deficiency in the fund, in which cases there was not any change in either of the rates per cent.

The above Statement XIX, A, will next include the present annual increment consisting of the income from the present investments prior to the variation occurring, and also the original annual instalment. Both these annual sums will be converted, by calculation at the future accumulation rate, into equivalent amounts of original loan repayable at the end of the unexpired period. The balance will represent the amount of original loan for which further provision has to be made caused by the decrease in the rates of income or of accumulation, and from this balance of loan the required additional annual instalment may be ascertained on standard calculation form, No. 3x. There is a difference in the method of treating the income from investments in Statements XX, A, and XXI, A, as compared with Statement XIX, A, but they may all be treated by the deductive method summarised at the head of this chapter. A further statement similar to XIX, B, is then prepared in each case showing how the fund will ultimately work out to repay the full amount of the loan at the end of the original repayment period.

Having ascertained the future or amended annual instalment in each case by the deductive method, the results will afterwards be used to derive therefrom a simple rule and formula by which to make the calculation by direct reference to the published tables or formulæ. It will then be found that by taking the present annual increment as the prime factor instead of the annual instalment, all such variations may be divided into two classes depending entirely upon the rate of accumulation. In variations similar to A and C, in which the rate of accumulation is reduced or increased, a calculation must be made by means of the tables or formula, but in variations similar to B, where there is a variation in the rate of income only, the rate of accumulation remaining unaltered, the amended annual instalment may be ascertained without calculation. This method is shown in Statement XX. C. called "the direct method," and it may appear superfluous to include the longer deductive method shown in Statement XX, A.

It is necessary, however, to state that in all cases the income from investments has been treated as being received annually, whereas in all probability it would be received half-yearly. The difference between an annual and a semi-annual accumulation has been pointed out at the end of Chapter V, giving also

the methods to be adopted in the case of any periodic accumulation other than annual.

The Annual Increment (Ratio) Method. It has already been stated that instead of using the above deductive method the same result may be obtained by direct calculation by means of a formula and rule. This will be fully described in Chapter XXII (Calculation XXII, C), which shows the future or amended annual increment of £1060·474 as found by the above deductive method. From the amended annual increment so found the future or amended annual instalment may be obtained by deducting therefrom the future annual income from the present investments. This ratio method by direct calculation will also apply to Variation C, where there is also a change in the rate of accumulation, but in the case of Variation B, in which the rate of accumulation remains unaltered, the amended annual instalment can be ascertained by a much more direct method without calculation, as explained above.

The method of finding the amended annual instalment is shown in the following Statement XIX, A, and the final repayment of the loan thereby is shown in Statement XIX, B, and in the pro forma account No. 7.

The Rate per cent. **Statement XIX. A.**

The Deductive Method.

Variation A, rate of accumulation only.

Showing the method of adjusting the annual instalment, in consequence of a variation in the rate of accumulation without any variation in the rate of income from the present investments, or in the period of repayment.

This example is compared with the conditions at the time the original calculation was made.

Conditions before adjustment (at end of 12th year)

Amount of loan repayable in 25 years	£26,495
Amount in the fund (at end of 12th year)	£9932·74
Present annual income (previously) received therefrom, at $3\frac{1}{2}$ per cent. per annum	£347·648
Present annual instalment, to be accumulated for 13 years at $3\frac{1}{2}$ per cent.	£680·234
Present annual increment	£1027·882

The Rate per cent. Statement XIX. B.

Variation A, rate of accumulation only.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the sinking fund after making the adjustment in the annual instalment, consequent upon a variation in the rate of accumulation, without any variation in the rate of income upon the present investments, or in the period of repayment.

		Equivalent amount of original loan.
Present investments (at end of 12th year)		£9932·74
Amended annual increment :—		
Original annual instalment	£680·234	
Additional annual instalment ...	32·592	
Total out of revenue	£712·826	
Income from investments	347·648	
	£1060·474	
Amount thereof, accumulated for 13 years at 3 per cent. Calculation (XIX) 4		£16562·26
Amount of original loan		£26495·00

Pro forma Sinking Fund Account, No. 7.

A Variation in the Rate of Accumulation only.

Loan of £26,495 repayable at the end of 25 years.

Showing the final repayment of the loan, by the operation of the increased annual instalment of £712·826.

Statement XIX. B. Rate of accumulation, 3 per cent.

Year.	Amount in the fund at beginning of year.	Income received from investments 3½ per cent.	Annual sinking fund instalment.	Income received from investments made after 12th year 3 per cent.	Amount in the fund at end of year.	Year.
1						1
2						2
3						3
4	The amount in the fund at the end of					4
5	the 12th year, £9932·744, is the correct					5
6	calculated amount, as shown by Calcula-					6
7	tion (XV) 2, and by the pro forma					7
8	account, No. 1, Chapter XV.					8
9						9
10						10
11						11
12					9932·744	12
13	9932·744	347·648	712·826	—	10993·218	13
14	10993·218	347·648	712·826	31·814	12085·506	14
15	12085·506	347·648	712·826	64·583	13210·563	15
16	13210·563	347·648	712·826	98·335	14369·372	16
17	14369·372	347·648	712·826	133·099	15562·945	17
18	15562·945	347·648	712·826	168·906	16792·325	18
19	16792·325	347·648	712·826	205·787	18058·586	19
20	18058·586	347·648	712·826	243·775	19362·835	20
21	19362·835	347·648	712·826	282·903	20706·212	21
22	20706·212	347·648	712·826	323·204	22089·890	22
23	22089·890	347·648	712·826	364·714	23515·078	23
24	23515·078	347·648	712·826	407·470	24983·022	24
25	24983·022	347·648	712·826	451·504	26495·000	25

CHAPTER XX.

SINKING FUND PROBLEMS, RELATING TO THE RATES PER CENT. OF INCOME AND ACCUMULATION (*Continued*).

VARIATION B, IN WHICH THERE IS A VARIATION IN THE RATE OF INCOME UPON THE PRESENT INVESTMENTS WITHOUT ANY VARIATION IN THE RATE OF ACCUMULATION OR PERIOD OF REPAYMENT. STATEMENT XX. A.

SUMMARY OF THE METHODS OF ADJUSTMENT. THE DEDUCTIVE METHOD. THE DIRECT METHOD WITHOUT CALCULATION. THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN BY THE OPERATION OF THE AMENDED ANNUAL INSTALMENT.

Summary of the methods of adjustment.

(I) The deductive method, as summarised at the head of Chapter XIX, will not apply, since there is not any variation in the rate of accumulation. The following adjustment by the deductive method is only of academic interest and has hardly any practical value. *Statement XX. A.*

(II) The direct method, without calculation, as summarised below, should always be used in actual practice.

Statement XX. C.

(III) The annual increment (balance of loan) method, as summarised at the head of Chapter XXII, may be used.

Statement XX. D.

(IV) The annual increment (ratio) method, as summarised at the head of Chapters XXIII, XXV, and XXVI, will not apply to this variation, as there is not any change in the rate of accumulation.

Note. The terms used in the following summary are fully explained at the head of Chapter XXII. If it be known or anticipated that the rate of income to be yielded in future by the present investments representing the fund will not be uniform during the whole of the unexpired portion of the repayment period the above methods will not apply, and the

adjustment must be made by the method fully described in Chapter XXVII.

SUMMARY OF THE DIRECT METHOD (*without calculation*), *of ascertaining the amended annual sinking fund instalment due to a variation in the rate of income yielded by the present investments without any variation in the rate of accumulation or in the period of repayment.* *Statement XX. C.*

(*1*) *Having ascertained the value of the present investments in the manner already described,*

(*2*) *Calculate the annual income previously received therefrom during the expired portion of the original repayment period (the present annual income).*

(*3*) *Calculate the annual income expected to be received therefrom during the unexpired portion of the original repayment period at the future rate per cent. of income (the future annual income).*

(*4*) *Ascertain the decrease or increase in such future annual income as compared with the annual income previously received.*

(*5*) *Add to, or deduct from, the original annual instalment the annual decrease or increase of income so ascertained.*

(*6*) *The result is the amended annual instalment to be set aside out of revenue or rate during the unexpired portion of the original repayment period.*

(*7*) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XX. B.*

(*8*) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired repayment period.*

Pro forma account, No. 8.

The amounts in the fund at the end of each year will be the same as in the original pro forma account since there is not any variation in the rate of accumulation or period of repayment, but the annual increment, although unaltered will have a different origin. Pro forma account No. 1, Chapter XV, will not apply in this case, the rate of accumulation being $3\frac{1}{2}$ per cent.

THE DEDUCTIVE METHOD. After discussing the deductive method of finding the amended annual instalment due to a change in the rate per cent. of accumulation (Variation A) in

Chapter XIX, a summary of the successive stages of the adjustment has been prepared and placed at the head of that chapter, and it is therefore only necessary to refer to that summary. Attention has already been drawn to the general considerations to be borne in mind in the rectification of a sinking fund in consequence of any variation in the rates per cent. of income or accumulation. The variation about to be considered is based upon the same imaginary sinking fund as Variation A (rate of accumulation), details of which are given in the previous chapter. At the end of the 12th year the sinking fund stands at the proper calculated amount of £9932·74, as found by Calculation (XV) 2. But whereas the conditions in Variation A (rate of accumulation) were compared with the original conditions, the present Variation B (rate of income), will be compared with the conditions in Variation A (rate of accumulation).

The rate of income is reduced from $3\frac{1}{2}$ to 3 per cent., but the rate of accumulation is unaltered, and remains at 3 per cent.

It has been stated in the previous chapter that the future rate of accumulation is the most important factor in the adjustment. That conclusion was based, in advance, upon the results of the discussion of the present variation, because, although the same deductive method will be used which has been applied to Variation A (rate of accumulation), this method is quite unnecessary in practice, although it is instructive as illustrating the predominant effect of the variation in the rate of accumulation.

It will be found that when the variation in the rate per cent. applies only to the rate of income from the present investments there is not any necessity to make any calculation whatever beyond adding to the original annual sinking fund instalment an amount equal to the annual loss of income caused by the reduced yield per cent. of the present investments, or by deducting therefrom any increase in such annual income. The remarks in the previous chapter, as to the three variations being derived by successive stages from the original conditions should be carefully remembered, and will be further emphasised.

The original and varied conditions are given in the following Statement, XX. A., and attention is again drawn to the fact that in this case also the income from investments is treated as being received annually, instead of semi-annually. Two statements will be prepared exactly similar in principle to those in the previous chapter, dealing with Variation A (rate of accumulation), showing in XX. A. the deductive method of ascertaining the amended annual instalment, and in XX. B.

the final repayment of the loan by the operation of the sinking fund, under the altered conditions. For the purpose of the comparison to be made later, this variation will also be compared with the original conditions. (See Statements XX. A. and XX. B. at end of chapter.)

THE DIRECT METHOD (without calculation). It has been pointed out in the previous chapter dealing with a variation in the rate of accumulation only that instead of making use of the deductive method, there described, for the purpose of ascertaining the amended annual instalment, the same result may be obtained by direct calculation by means of a rule and formula, which will be fully described in Chapter XXIII, namely, the annual increment (ratio) method. This remark applied to Variation A as compared with the original conditions in which there is a reduction in the rate of accumulation, but without any variation in the rate of income from investments. In the present case, Variation B, as compared with the conditions in Variation A (rate of accumulation) there is a reduction in the rate of income upon the present investments, without any variation in the rate of accumulation, and the deductive method will again be used. On comparing the two results, it is found that in both cases the future or amended annual increment is £1060·474, although the amended annual instalment is increased, namely, from £712·826 in Variation A to £762·490 in Variation B. The difference between the two amended annual instalments is £49·664, which is the amount by which the future annual income in Variation A is reduced owing to the fall of ½ per cent. in the rate of income to be yielded by the present investments under the altered conditions of Variation B, namely, from £347·648 in Variation A to £297·984 in Variation B.

This proves that when the rate of accumulation remains unaltered, there is not any alteration in the annual increment, and, further, that the amended annual instalment may be ascertained without any calculation whatever, by merely adding to the present annual instalment the amount of the decrease in the annual income to be received from the present investments under the altered conditions, and the same applies equally to an increase in the rate per cent. yielded by the present investments.

The following Statement XX. C. illustrates the adjustment by the direct method, without calculation.

Although the direct method of finding the amended annual

instalment will be sufficient in all cases where there is not any variation in the rate of accumulation, it should be proved by preparing a statement similar to No. XX. B. showing the position of the fund and the final repayment of the loan after making the adjustment. The rule and formula to be described later in Chapter XXIII (the annual increment (ratio) method), by which the future or amended annual increment under the altered conditions may be found by direct calculation from the present annual increment under the previous conditions, cannot obviously be applied to cases in which there is not any variation in the annual increment, which depends entirely upon the rate of accumulation.

In the previous chapter the deductive method is employed to ascertain the amended annual instalment, consequent upon a variation in the rate of accumulation only. In the following Chapter (XXI), in discussing Variation C, it will be seen that this deductive method is also available for ascertaining the amended annual instalment consequent upon a variation in the rate of accumulation, accompanied by a variation in the rate of income from the present investments. But in the case of a variation in the rate of income only, the deductive method may be replaced by one much simpler. At the head of this chapter, therefore, although reference is made to the deductive method as summarised in Chapter XIX, the direct method without calculation has been treated as the standard method to be adopted in practice, and has been stated in summary form.

In Chapter XIX, the conditions in Variation A (rate of accumulation) are compared with the original conditions, and it has been found that an additional annual instalment of £32·592 is required to compensate for the decrease in the rate of accumulation. Proceeding to Variation B, it has been found that although the rate of accumulation remains unaltered, the rate of income from investments is reduced. This reduction in income requires a further increase in the annual instalment of £49·664. It is now possible to compare the amended annual instalment in Variation B, with the annual instalment under the original conditions as follows:—

The original annual instalment was		£680·234
Additional instalment due to the reduction in the rate of accumulation.	Variation A.	32·592
Additional instalment due to the reduction in the rate of income from investments.	Variation B	49·664
Amended annual instalment.	Variation B	£762·490

or an increase of £82·256, but on comparing the annual increment in Variation B (rate of income), with the annual increment under the original conditions, it is increased by only £32·592, namely, from £1027·882 to £1060·474. This further proves that so long as the rate of accumulation remains unaltered the annual increment does not require to be amended, but if the portion of the annual increment derived from outside investments is reduced, owing to a fall in the rate of income yielded by the present investments, the burden must be borne by the other partner, namely, the revenue or rate account which provides the annual instalment.

Statement XX. D. shows the method of making the adjustment by the annual increment (balance of loan) method, which will be fully described and summarised in Chapter XXII.

The Rate per cent. Statement XX. A.

The Deductive Method.

Variation B, rate of income only.

Showing the method of adjusting the annual instalment in consequence of a variation in the rate of income upon the present investments without any variation in the rate of accumulation or in the period of repayment.

This example is compared with the original conditions as modified by Variation A.

Conditions before adjustment (at end of 12th year),

Amount of loan repayable in 25 years	£26,495
Amount in the fund (at end of 12th year)	£9932·74
Present annual income (previously) received therefrom, at $3\frac{1}{2}$ per cent. per annum	£347·648
Present annual instalment, to be accumulated for 13 years at 3 per cent.	£712·826
Present annual increment	£1060·474

Variation from the above conditions :—

The rate of income yielded by the present investments is reduced from $3\frac{1}{2}$ to 3 per cent.

Future annual income	£297·984
Reduction in annual income	49·664
Increased annual instalment	49·664
Future annual increment	1060·474

R

		Equivalent amount of original loan.
Present investments (at end of 12th year), representing the amount now in the fund ...		£9932·74
Future annual income from present investments :—		
Amount of an annuity of	£297·984	
accumulated for 13 years, at 3 per cent. Calculation (XX) 1		£4653·85
Original annual instalment :—		
Amount of an annuity of	£680·234	
accumulated for 13 years, at 3 per cent. Calculation (XIX) 2		£10623·75
Additional annual instalment (*Variation A*) :—		
Amount of an annuity of	£32·592	
accumulated for 13 years, at 3 per cent. Calculation (XIX) 3		£509·02
Provision already made will repay loan of		£25719·36
Additional annual instalment required :—		
Balance, being amount of original loan unprovided for owing to the above decrease in the rate of income from investments requiring an additional annual instalment, to be set aside and accumulated for 13 years at 3 per cent.		£775·64
Additional annual instalment Calculation (XX) 2	£49·664	
Amount of original loan		£26495·00

Amended annual increment, *being* :—

Income from investments	£297·984
Amended annual instalment	762·490
	£1060·474

The Rate per cent. Statement XX. B.

Variation B, rate of income only.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the sinking fund after making the adjustment in the annual instalment, consequent upon a variation in the rate of income upon the present investments without any variation in the rate of accumulation, or in the period of repayment.

			Equivalent amount of original loan.
Present investments (at end of 12th year) ...			£9932·74
Amended annual increment:—			
Original annual instalment		£680·234	
Additional.	Variation A	32·592	
ditto.	Variation B	49·664	
Total out of revenue		£762·490	
Income from investments		297·984	
		£1060·474	
Amount thereof, accumulated for 13 years at 3 per cent.	Calculation (XX) 3		£16562·26
Amount of original loan			£26495·00
Amended annual instalment ...		£762·490	

The Rate per cent. Statement XX. C.

The Direct Method (without calculation).

Variation B, rate of income only.

Showing the method of adjusting the annual instalment in consequence of a variation in the rate of income upon the present investments without any variation in the rate of accumulation or in the period of repayment.

Required the amended annual instalment, to be set aside and accumulated as a sinking fund to compensate for a reduction, from $3\frac{1}{2}$ to 3 per cent., in the rate of income to be received from the present investments, valued at £9932·74. Rate of accumulation 3 per cent.

Annual sinking fund instalment, at date of adjustment as calculated or as ascertained in Variation A. Statement XIX. B.		£712·826
Add decrease in annual income from investments		
at $3\frac{1}{2}$ per cent.	£347·648	
at 3 per cent.	297·984	
		£49·664
Amended annual instalment		£762·490

Memo. In the case of an increase in the amount of the future annual income, such increased income should be deducted from the original annual instalment.

The Rate per cent. Statement XX. D.

The Annual Increment (balance of loan) Method.

Variation B, rate of income only.

To find the amended annual sinking fund instalment consequent upon a variation in the rate of income upon the present investments, without any variation in the rate of accumulation, or in the period of repayment.

Rate of income from investments reduced from $3\frac{1}{2}$ to 3 per cent.

Rate of accumulation, 3 per cent.

For Rule, see Chapter XXII.

Amount of original loan (25 years)		£26495·00
deduct amount in the fund at the end of the 12th year		£9932·74
Balance of loan		£16562·26

Amended annual increment, to be added to the fund, and accumulated at 3 per cent., to provide this amount at the end of 13 years Calculation (XX) 4	£1060·474
deduct income to be received from the present investments (£9932·74) at 3 per cent.	£297·984
Amended annual instalment	£762·490

Pro forma Sinking Fund Account, No. 8.

A Variation in the rate of Income upon the present Investments.

Loan of £26,495, repayable at the end of 25 years.

Showing the final repayment of the loan, by the operation of the increased annual instalment of £762·490.

Statement XX. B. Rate of accumulation, 3· per cent.

Year.	Amount in the fund at beginning of year.	Income received from investments.	Annual sinking fund instalment.	Income received from investments made after 12th year 3 per cent.	Amount in the fund at end of year.	Year.
1						1
2						2
3						3
4		The amount in the fund at the end of the 12th year, £9932·744, is the correct calculated amount, as shown by Calculation (XV) 2, and by the pro forma account, No. 1, Chapter XV.				4
5						5
6						6
7						7
8						8
9						9
10						10
11						11
12					9932·744	12
13	9932·744	297·984	762·490	—	10993·218	13
14	10993·218	297·984	762·490	31·814	12085·506	14
15	12085·506	297·984	762·490	64·583	13210·563	15
16	13210·563	297·984	762·490	98·335	14369·372	16
17	14369·372	297·984	762·490	133·099	15562·945	17
18	15562·945	297·984	762·490	168·906	16792·325	18
19	16792·325	297·984	762·490	205·787	18058·586	19
20	18058·586	297·984	762·490	243·775	19362·835	20
21	19362·835	297·984	762·490	282·903	20706·212	21
22	20706·212	297·984	762·490	323·204	22089·890	22
23	22089·890	297·984	762·490	364·714	23515·078	23
24	23515·078	297·984	762·490	407·470	24983·022	24
25	24983·022	297·984	762·490	451·504	26495·000	25

CHAPTER XXI.

SINKING FUND PROBLEMS RELATING TO THE RATES PER CENT. OF INCOME AND ACCUMULATION (*Continued*).

VARIATION C, IN WHICH THERE IS A VARIATION IN THE RATE OF ACCUMULATION AND ALSO IN THE RATE OF INCOME UPON THE PRESENT INVESTMENTS, BUT WITHOUT ANY VARIATION IN THE PERIOD OF REPAYMENT.

SUMMARY OF THE METHODS OF ADJUSTMENT. THE DEDUCTIVE METHOD. COMPARISON OF RESULTS OBTAINED IN THIS AND PREVIOUS CHAPTERS IN ALL PROBLEMS INVOLVING A VARIATION IN THE RATE PER CENT. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN BY THE OPERATION OF THE AMENDED ANNUAL INSTALMENT.

Summary of the methods of adjustment.

(I) The deductive method, as summarised at the head of Chapter XIX, will apply, but has been slightly modified in Statement XXI. A.

(II) The direct method, without calculation, as summarised at the head of Chapter XX, will not apply to this variation.

(III) The annual increment (balance of loan) method, as summarised at the head of Chapter XXII, may be used, but will not be applied to the example under review.

(IV) The annual increment (ratio) method, as summarised at the head of Chapter XXIII, may be used, but will not be applied to the example under review.

Note. The terms used in the summaries above mentioned are fully explained at the head of Chapter XXII. In all the above methods it is imperative that the rates of accumulation and of income from investments be uniform during the whole of the unexpired or substituted period of repayment.

The enquiry into the methods of adjusting the annual sinking fund instalment in consequence of any variation in the rate per cent. is now almost completed. Variation A, which affected the rate of accumulation only, is fully discussed in Chapter XIX. In the case of Variation B (Chapter XX) the varying factor is the rate per cent. of income to be yielded by the present investments representing the fund. The enquiry will now be completed by examining Variation C, in which there is a simultaneous change in both the above rates, and the deductive method will again be used, as fully described in Chapter XIX, and of which a summary of the various stages is placed at the beginning of that chapter. The two preceding chapters deal exhaustively with all general questions affecting the enquiry, and they will apply equally to the present variation.

The position of the fund at the time of making the adjustment is fully set out in the following Statement XXI. A., which is similar to those prepared to illustrate the variations already considered. These conditions are based upon those obtaining when the original calculation was made, and although the present example will be compared with the conditions in Variation A (rate of accumulation) they will also be compared with those originally existing. As in previous instances it will be assumed that all sums are added to the fund and accumulated annually. In the following chapter (XXII) the whole of the results of the enquiry into the rate per cent. will be compared in order to show the general effect of such rate upon the accumulation of a sinking fund. The investigation will then be extended in order to derive a rule and formula by means of which the adjustments may be made by the more direct annual increment (ratio) method.

Similar statements have been prepared as in the previous variations, namely, XXI. A., showing the amended annual instalment as ascertained by the deductive method; and XXI. B., showing the final repayment of the loan by the operation of the amended annual instalment so ascertained, as shown by the pro forma account, No. 9.

The results already obtained may now be briefly stated, both with regard to the original conditions as well as Variations A, B, and C, in order to show the progressive variations in the examples which will be used later in discussing the derivation of a rule and formula which may be applied to any problem relating to the rate per cent.

The sinking fund instalment as originally calculated, at a rate of accumulation of 3½ per cent., was

Calculation (XV) 1	£680·234
In Variation A (rate of accumulation only), as compared with the original conditions, the rate of income from investments remained unaltered, namely, 3½ per cent., but the rate of accumulation was reduced from 3½ to 3 per per cent., requiring an additional instalment, as shown by Calculation (XIX) 3 of	£32·592
Amended instalment (A), Statement XIX. A.	£712·826
In Variation B (rate of income only) as compared with Variation A, the rate of accumulation remained at 3 per cent., but the rate of income from investments was reduced from 3½ to 3 per cent., requiring an additional instalment as shown by Calculation (XX) 2 of	£49·664
Amended instalment (B), Statement XX. A.	£762·490
In Variation C, as compared with Variation A, the rate of accumulation was further reduced from 3 to 2½ per cent., and the rate of income from investments was also reduced from 3½ to 3 per cent. This required an additional instalment, as shown by Calculation (XXI) 4, of £83·099 but part of this was due to the reduction in the rate of income in Variation B, as above ... ... £49·664	£33·435
Amended instalment (C.) Statement XXI. A.	£795·925

which amended annual instalment is required to be set aside out of revenue or rate and accumulated in addition to the income from the present investments, in order to provide the loan repayable at the end of the prescribed period.

In Statement XXI. A. following, the additional annual instalment is ascertained to be £83·099. This annual increase is derived directly from the conditions in Variation A, and is made up of the increased instalment due to the reduction in the

rate of income in Variation B, viz., £49·664, and the above amount of £33·435 due to the variation in the rate of accumulation. In the present example there is a variation in both the rates of income and of accumulation, and in this respect it combines the changes in Variations A and B. In Variation B, where there is a change in the rate of income only, the annual instalment is corrected by adding thereto the actual deficiency in the future annual income. Statement XXI. A. shows by the deductive method that the amount of original loan which would be unprovided in consequence of the concurrent reduction in the above rates is £1258·15, requiring an additional annual instalment of £83·099, as shown by Calculation (XXI) 4. This additional annual instalment is the measure of the two annual losses of interest, and it is possible to allocate to each rate the proportions in which they contribute thereto. This will be seen by referring to the following Statement XXI. C., in which column (2) contains, in the case of Variation A, the original and additional annual instalments, and also the income from investments, making up the amended annual increment. This agrees with Statement XIX. A. in total, but the income from investments at 3½ per cent. has been divided as between 3 and ½ per cent. in order to compare this variation with Variation C. The final column (8) shows the deficiency of original loan caused by the accumulation of each of the component parts of the annual increment at 2½ per cent. in Variation C, instead of at 3 per cent., as in Variation A.

This deficiency is arrived at by deducting the amount of loan in column (7) from the amount in column (4), and is made up as follows, expressed in terms of original loan:—

Deficiency due to the reduction in the rate of accumulation, from 3 to 2½ per cent., of items 1 to 3	£482·51
Deficiency due to the reduction in the accumulation of the decrease in income, item 4	£23·70
	£506·21
Deficiency of annual income accumulated at 2½ per cent.	£751·94
	£1258·15

The deficiency of £1258·15 of loan requires a total additional annual instalment of £83·099, as previously ascertained, which is made up of:—

the loss of income from investments	£49·664
and the loss owing to the reduction in the rate of accumulation	£33·435
	£83·099

The above amount of £33·435 includes the loss of accumulation not only upon the remaining portion (£1010·810) of the present annual increment in Variation A, as shown in column 2, but also upon the reduction in the annual income, viz., £49·664. This proves that when, as in Variation C, the reduction in the rate of income from investments is accompanied by a reduction in the rate of accumulation, the additional annual instalment is measured, not by the actual reduction in the annual income, as in Variation B, but by the annual deficiency of income increased in the ratio that the amount of £1 per annum at the past rate bears to the amount of £1 per annum at the future rate, in each case for the same number of years, being the unexpired portion of the original repayment period. This will be referred to later in Chapter XXII, when discussing Calculation (XXII) E. with the object of arriving at a method of making the adjustment by the more direct annual increment (ratio) method. In that case the comparison will be made between Variation C and the original conditions, but the same principles apply, and the above table may be again referred to with advantage. (Statement XXI. C. follows.)

The Rate per cent. Statement XXI. A.

The Deductive Method.

Variation C, rates of accumulation and income combined.

Showing the method of adjusting the annual instalment in consequence of a variation in the rate of accumulation and also in the rate of income upon the present investments, but without any variation in the period of repayment.

This example is compared with the original conditions as modified by Variation A.

Conditions before adjustment (at end of 12th year)

Amount of loan repayable in 25 years	£26,495
Amount in the fund (at the end of 12th year) ...	£9932·74
Present annual income (previously) received therefrom, at $3\frac{1}{2}$ per cent. per annum	£347·648
Present annual instalment, to be accumulated for 13 years at 3 per cent.	£712·826
Present annual increment	£1060·474

Variation from the above conditions:—

The rate of accumulation of the fund is reduced from 3 to $2\frac{1}{2}$ per cent.

The rate of income yielded by the present investments is reduced from $3\frac{1}{2}$ to 3 per cent.

Future annual income	£297·984
The future rate of accumulation	$2\frac{1}{2}$ per cent.

Present investments (at end of 12th year), representing the amount now in the fund ...		£9932·74
Future annual income from present investments:—		
Amount of an annuity of	£297·984	
accumulated for 13 years, at $2\frac{1}{2}$ per cent. Calculation (XXI) 1		£4511·61
Original annual instalment:—		
Amount of an annuity of	£680·234	
accumulated for 13 years, at $2\frac{1}{2}$ per cent. Calculation (XXI) 2		£10299·04
Additional annual instalment (*Variation A*):—		
Amount of an annuity of	£32·592	
accumulated for 13 years, at $2\frac{1}{2}$ per cent. Calculation (XXI) 3		£493·46
Provision already made will repay loan of		£25236·85
Additional annual instalment required:—		
Balance, being amount of original loan unprovided for owing to the above decrease in the rate of accumulation, and in the rate of income from investments requiring an additional annual instalment, to be set aside and accumulated for 13 years at $2\frac{1}{2}$ per cent.... ...		£1258·15
Additional annual instalment Calculation (XXI) 4	£83·099	
Amount of original loan		£26495·00
Amended annual increment, *being*:—		
Income from investments	£297·984	
Amended annual instalment	795·925	
	£1093·909	

The Rate per cent. | Statement XXI. B.

Variation C, rates of accumulation and income

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the sinking fund after making the adjustment in the annual instalment, consequent upon a variation in the rate of accumulation, and also in the rate of income upon the present investments, but without any variation in the period of repayment.

			Equivalent amount of original loan.
Present investments (at end of 12th year) ...			£9932·74
Amended annual increment :—			
Original annual instalment		£680·234	
Additional.	Variation A	32·592	
ditto.	Variation C	83·099	
Total out of revenue		795·925	
Income from investments		297·984	
		£1093·909	
Amount thereof, accumulated for 13 years at $2\frac{1}{2}$ per cent.	Calculation (XXI) 5		£16562·26
Amount of original loan			£26495·00

Amended annual instalment ... £795·925

The Rate per cent.

Statement XXI, C.

Variation A. Rate of accumulation. Variation C. Rates of accumulation and income from investments.

Showing the composition of the additional annual instalment in Variation C as compared with Variation A, distinguishing between the portions thereof due to the reduction in the rates of accumulation and income respectively.

	Variation A. Rate of Accumulation 3 per cent.			Variation C. Rate of Accumulation $2\frac{1}{2}$ per cent			
Annual Increment. (1)	Annual increment. (2)	Calculation. (3)	Original loan. (4)	Annual increment. (5)	Calculation. (6)	Original loan. (7)	Deficiency of original loan. (8)
1. Income from investments, 3 per cent. ...	297·984	(XX) 1	4653·85	297·984	(XXI) 1	4511·61	142·24
2. Original instalment	680·234	(XIX) 2	10623·75	680·234	(XXI) 2	10299·04	324·71
3. Additional instalment, Variation A	32·592	(XIX) 3	509·02	32·592	(XXI) 3	493·46	15·56
	1010·810	—	15786·62	1010·810	—	15304·11	482·51
Additional annual instalment due to:—							
4. Decreased income from investments, $\frac{1}{2}$% ...	49·664	(XX) 2	775·64	49·664	(XXI) 4	751·94	23·70
5. Decrease in rate of accumulation	—	—	—	33·435	(XXI) 4	506·21	506·21
							751·94
Amended annual increment	1060·474	XIX. B.	16562·26	1093·909	XXI. B.	16562·26	1258·15
Amended annual instalment:—							
Original, $3\frac{1}{2}$ per cent.	680·234	(XIX) 2	10623·75	680·234	(XXI) 2	10299·04	
Additional, Variation A	32·592	(XIX) 3	509·02	32·592	(XXI) 3	493·46	
Decreased income, Variation B...	—	—	—	49·664	(XXI) 4	751·94	
Reduced accumulation, Variation C ...	—	—	—	33·435	(XXI) 4	506·21	
	712·826	—	11132·77	795·925	—	12050·65	
Income from investments	347·648	(XIX) 1	5429·49	297·984	—	4511·61	
Amount in the fund	—	—	9932·74	—	—	9932·74	
Amount of original loan	1060·474	XIX. B.	26495·00	1093·909	—	26495·00	

Pro forma Sinking Fund Account, No. 9.

A Variation in the rate of Accumulation, as well as in the Rate of Income upon the present Investments.

Loan of £26,495, repayable at the end of 25 years.

Showing the final repayment of the loan, by the operation of the increased annual instalment of £795·925.

Statement XXI. B. Rate of Accumulation, $2\frac{1}{2}$ per cent.

The amount in the fund at the end of the 12th year, £9932·744, is the correct calculated amount, as shown by Calculation (XV) 2, and by the pro forma account, No. 1, Chapter XV.

Year.	Amount in the fund at beginning of year.	Income received from investments.	Annual Sinking Fund instalment	Income received from investments $2\frac{1}{2}$ per cent.	Amount in the fund at end of year.	Year.
1						1
2						2
3						3
4						4
5						5
6						6
7						7
8						8
9						9
10						10
11						11
12					9932·744	12
13	9932·744	297·984	795·925	—	11026·653	13
14	11026·653	297·984	795·925	27·348	12147·910	14
15	12147·910	297·984	795·925	55·379	13297·198	15
16	13297·198	297·984	795·925	84·111	14475·218	16
17	14475·218	297·984	795·925	113·562	15682·689	17
18	15682·689	297·984	795·925	143·749	16920·347	18
19	16920·347	297·984	795·925	174·690	18188·946	19
20	18188·946	297·984	795·925	206·405	19489·260	20
21	19489·260	297·984	795·925	238·913	20822·082	21
22	20822·082	297·984	795·925	272·233	22188·224	22
23	22188·224	297·984	795·925	306·387	23588·520	23
24	23588·520	297·984	795·925	341·394	25023·823	24
25	25023·823	297·984	795·925	377·268	26495·000	25

Section IV.

Sinking Fund Problems

The Annual Increment

CHAPTER XXII.

THE ANNUAL INCREMENT METHODS.

DEFINITION OF TERMS RELATING TO THE ANNUAL INCREMENT AND THE METHODS OF ASCERTAINING THE AMENDED ANNUAL INSTALMENT BASED THEREON. THIS APPLIES TO ALL VARIATIONS IN THE RATE OF ACCUMULATION AND THE PERIOD OF REPAYMENT, WITH OR WITHOUT ANY VARIATION IN THE RATE OF INCOME UPON THE PRESENT INVESTMENTS REPRESENTING THE FUND.

SINKING FUND PROBLEMS RELATING TO THE RATE PER CENT. OF ACCUMULATION.

METHODS OF ASCERTAINING THE AMENDED ANNUAL INSTALMENT BY DIRECT CALCULATION IN TERMS OF THE ANNUAL INCREMENT. COMPARISON OF THE RESULTS ALREADY OBTAINED IN CHAPTERS XIX, XX, AND XXI IN TERMS OF THE ANNUAL INSTALMENT WITH THOSE OBTAINED BY MEANS OF THE ANNUAL INCREMENT AND THE VARYING RATES OF ACCUMULATION. THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD.

Summary of the methods of adjustment.

(I) The deductive method, as summarised at the head of Chapter XIX,

as to the rate of accumulation, *Statement XIX. A.*

as to the rate of income and the rate of accumulation, *Statement XXI. A.*

(II) The direct method, without calculation, as summarised at the head of Chapter XX, will not apply to these variations.

(III) The annual increment (balance of loan) method, as summarised below, is illustrated in the text.

(IV) The annual increment (ratio) method, as summarised at the head of Chapter XXIII, *Statement XXII. C.*

Note. In all cases where the adjustment is made by the annual increment methods it is imperative that the rates per cent., both of accumulation and income from investments, be uniform during the whole of the unexpired or substituted period of repayment.

The Annual Increment Methods, Definition of Terms.

The present annual increment, at the time of making an adjustment in the annual instalment, consequent upon a variation in the rate of accumulation, or in the period of repayment, or in both these factors combined, is composed of:

1. *The present or original annual instalment, which has been set aside and added to the sinking fund up to the time of making the adjustment, and*

2. *The present annual income from investments, representing the fund, which has been received up to the date of making the adjustment.*

The future or amended annual increment, consequent upon a variation in either or both of the factors of rate per cent. of accumulation and period of repayment, is composed of:—

1. *The future or amended annual instalment, required to be set aside and added to the sinking fund in consequence of the above variation or variations, and*

2. *The future annual income from investments, representing the fund at the date of making the adjustment whether the rate of income upon such investments remains unaltered, or is varied.*

The annual increments, as above described, are the primary and final factors in all the adjustments by these methods.

The past rate denotes the rate of accumulation upon which is based the present or original annual instalment included in the present annual increment.

The future rate, denotes the rate of accumulation to be used instead of the past rate to calculate the future or amended annual increment. It will be the same as the past rate in problems involving a variation in the period of repayment only, without any variation in the rate of accumulation.

The unexpired period, denotes the unexpired portion, at the time of making the adjustment, of the original repayment period upon which the present or original annual instalment was based.

The substituted period, denotes the increased or reduced number of years over which the future or amended annual instalments shall be spread and at the end of which the full amount of the loan will be repayable. It will be the same as the unexpired period in problems involving a variation in the rate of accumulation only, without a variation in the period of repayment.

The income from investments, representing the amount in the fund does not enter into the actual calculation except as a component part of the present and future or amended annual increments, as above defined.

The future or amended annual instalment, is obtained in all cases by deducting, from the ascertained amended annual increment, the future annual income from the present investments representing the fund, whether the rate of income upon such investments remains unaltered or is varied.

Note. The foregoing definitions will be referred to in subsequent chapters, without any further explanation or amplification.

General summary of the annual increment (balance of loan) method, *of ascertaining the amended annual sinking fund instalment due to a variation in either the rate of accumulation, the period of repayment, the rate of income upon the present investments representing the fund, or any of these factors in combination. The terms used in the following summary are fully explained above.*

(1) *Ascertain the value of the present investments in the manner already described, and deduct the value so obtained from the amount of the original loan.*

(2) *The remainder represents the balance of loan to be provided by the accumulation of the future or amended annual increment, as previously defined, for the unexpired or substituted repayment period at the future rate of accumulation.*

(3) *Calculate the annuity, or annual increment, to be added to the fund and accumulated for the period and at the rate per cent. as in* (2).

(4) *From the amended annual increment ascertained as in* (3), *deduct the future annual income to be received from the present investments during the whole of the unexpired or substituted repayment period.*

(5) *The remainder will represent the future or amended annual instalment to be charged to revenue or rate account, and added to the fund, during the whole of the unexpired or substituted repayment period.*

(6) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.*

(7) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the unexpired or substituted repayment period.*

Memo. In the event of the income from investments not being uniform over the whole of the repayment period, proceed by the method in Chapter XXVII.

SINKING FUND PROBLEMS, RELATING TO THE RATE PER CENT. OF ACCUMULATION. THE ANNUAL INCREMENT. In previous chapters dealing with the three variations in the rates per cent. of accumulation and income from the present investments, the amended annual instalment has been ascertained by the deductive method described in Chapter XIX.

This method is based upon (1) the value of the present investments representing the fund as described in Chapter XIV; (2) the annual income to be received therefrom, and (3) the original annual instalment. All these factors have been reduced to equivalent amounts of original loan ultimately repayable, in order to ascertain the deficiency in the fund at the end of the repayment period due to the reduction in the rate per cent. of income or of accumulation.

This deficiency of original loan ultimately repayable has been converted into an equal annual sinking fund instalment, to be provided out of revenue or rate, in addition to the original instalment. A statement has been prepared showing in each case the final repayment of the loan by the operation of the amended annual instalment so ascertained at the end of the 12th year.

In these statements the amount of the loan has been divided into two parts; the first (£9932·74) being the value of the present investments representing the fund, and the second (£16562·26) being the amount of loan to be provided at the end of the repayment period by the accumulation of the future or amended annual increment, which consists of:—

1. Income from the present investments.
2. The original annual instalment.
3. The additional annual instalment ascertained in the above manner,

thereby proving the accuracy of the results obtained by the deductive method. But the original annual instalment is the only constant factor, although it may in future accumulate at a lower rate than was originally estimated. Consequently, in arriving at the future or amended annual instalment two variable factors have to be considered, namely, (1) the rate of income upon the present investments, and (2) the rate of accumulation. These two factors of rate per cent. are most important in the after consideration of the problem because they may vary in different directions and are not in any way related. But any difficulty may be eliminated by treating the amount of the future annual income to be received from the present investments, at the future rate per cent. of income, as an annuity certain in the same manner as the original annual instalment. These two factors together constitute the future or amended annual increment of the fund, which is acted upon by the future rate of accumulation only, consequently the problem has been reduced to an annuity certain for a definite term at a given rate per cent. The annual income from the present investments, included in the present annual increment in all adjustments made by this method, is the annual amount which has been received in the past and is not the future annual income which will be yielded during the unexpired or substituted period of repayment. This is one of the fundamental principles of the annual increment (ratio) method. The enquiry is thereby transferred from the annual instalment to the annual increment, and as this is an annuity of fixed amount it is possible to arrive at a formula, and a rule based thereon. The annual increment has been fully described in Chapter XIV.

In Chapters XIX, XX, and XXI, three variations in the rate per cent. have been considered, and the amended annual instalment in each case has been ascertained by the deductive method. Up to this point the examples have been considered only as individual problems, but they will now be treated in

combination. In order, however, to avoid undue reference to previous chapters, the following Statement XXII. A. has been prepared containing the whole of the conditions in each case and the actual results previously obtained.

A further classified Statement XXII. B. has been prepared showing the initial conditions in each case and giving references to methods and calculations by which the results have been obtained. It should again be mentioned that, although in each variation a reduction has been assumed in the rates per cent., as being more likely to occur in practice, yet the same principles and methods will apply equally to an increase in both rates per cent., or to an increase in one rate and a reduction in the other.

Statement XXII. A. (page 265) contains full details of the amended annual instalments found by the deductive method in the three Variations A, B, and C, which are derived one from the other and from the original conditions by gradual variations in the rates of income and accumulation. There is therefore a definite relation between the original annual increment of £1027·882 and the successive annual increments in Variations A, B, and C, leading to the final annual increment of £1093·909 in Variation C. This relation depends upon the respective rates of accumulation in the four examples, and by this means it is possible to derive the rule and formula required. Statement XXII. B. contains the annual increments only, and shows the rates per cent. of income and accumulation in each case. All these annual sums are derived from a common source, and therefore may be treated as simple annuities for a term without reference to any principal sum or other factor than the rate of accumulation. In Statement XXII. B. the variations in the rate per cent. are divided into two classes depending upon the rate of accumulation. The first class contains the problems in which the rate of accumulation remains unaltered, and there is not therefore any necessity to sub-divide the class as regards any variation in the rate of income on the investments, because, as ascertained in considering Variation B, there is not any question of compound interest involved. It is only necessary to correct the original annual instalment by adding to or deducting therefrom the difference between the annual amounts of income yielded by the present investments at the past and future rates respectively. The second class includes cases in which there is a variation in the rate of accumulation, and this class may be sub-divided according as the rate of income upon the present investments is unaltered or is varied. Although it

will be found that both sub-divisions of this class may be treated by one and the same rule and formula, the present distinction is useful in giving emphasis to the fact. It will also be seen, in dealing with problems in which there is a change in the rate of income upon investments as well as in the rate of accumulation, Chapter XXI, that the reason why the rule applies is not so obvious as in the case of a simple variation in the rate of accumulation only.

Class I. *Variations in the rate of income from investments only, the rate of accumulation remaining unaltered.*

Problems of this nature, in which the variation affects only the rate of income on the present investments, but in which the rate of accumulation remains the same, have been fully described in Chapter XX, Variation B. The calculation of the amended annual instalment in such cases may be made by the deductive method, Statement XX. A., which applies equally to all manner of variations in the rate per cent. But Statement XX. C. shows that the amended annual instalment may be arrived at by a simple direct method, without calculation, although the deductive method may be used to prove the accuracy of the conclusions.

Class II. *Variations in the rate of accumulation.*

This class has been sub-divided into two groups, as shown in Statement XXII. B. as follows:—

(*A*) In which the rate of income upon the present investments is unaltered.

(*B*) In which the rate of income upon the present investments is varied.

Each of these sub-divisions will be considered in detail, taking as examples the figures given in Statement XXII. B.

The Rate per cent. Statement XXII, A.

Variation A. Rate of accumulation only. Chapter XIX.
Variation B. Rate of income only. Chapter XX.
Variation C. Rates of accumulation and income combined. Chapter XXI.

Showing, at the end of the 12th year, under the original conditions, and under each variation:—

(1) The present, and future or amended annual increments.
(2) The additional annual instalment distinguishing between the loss of income from the present investments, and the reduction in the rate of accumulation.
(3) The provision of the future annual increment from internal and external sources.

Loan £26,495. Amount in the fund at end of 12th year, £9,932·74.	Original Conditions	Variation A	Variation B	Variation C
Future rate of accumulation	$3\frac{1}{2}$	3	3	$2\frac{1}{2}$
Future yield on present investments	$3\frac{1}{2}$	$3\frac{1}{2}$	3	3
I. *Present annual increment*:—				
Original instalment	680·234	680·234	680·234	680·234
Income from present investments at end of 12th year, at above rates	347·648	347·648	297·984	297·984
Present annual increment which will continue to be accumulated at reduced rate of accumulation	1027·882	1027·882	978·218	978·218
Additional annual instalments to make good the loss of interest on present investments and future accumulations, to be added to the original annual instalments and provided out of revenue or rate	Nil	32·592	82·256	115·691
Future annual increment	1027·882	1060·474	1060·474	1093·909
II. The above additional annual instalments, as compared with the original conditions are made up as follows:—				
1. Decrease in income from the present investments ...	Nil	Nil	49·664	49·664
2. Decrease in interest on future accumulations due to reduction in rate of accumulation:—				
Variation A	—	32·592	32·592	32·592
Variation C	—	—	—	33·435
	Nil	32·592	82·256	115·691
III. *Future annual increment to be provided as follows*:—				
A. To be taken out of revenue or rate:—				
Original annual instalment	680·234	680·234	680·234	680·234
Deficiency in future income from present investments	—	—	49·664	49·664
Additional annual instalment to compensate for decrease in rate of accumulation ...	—	32·592	32·592	66·027
	680·234	712·826	762·490	795·925
B. Income to be received in future from present investments	347·648	347·648	297·984	297·984
Future annual increment	1027·882	1060·474	1060·474	1093·909

The Rate per cent.

Statement XXII. B.

Showing the respective annual increments under the original conditions as compared with the same under the several Variations A, B, and C, which will be referred to in the following chapters.

Class.	Calculation No.	Method of Calculation.	Original Conditions. Variation.	Original Rate. Inc.	Original Rate. Acc.	Original Annual Increment.	Amended Conditions. Variation	Amended Rate. Inc.	Amended Rate. Acc.	Amended Annual Increment.
I.	*Rate of accumulation remains unaltered.*									
	XX. B.	Deductive	A	$3\frac{1}{2}$	3	1060·474	B	3	3	1060·474
	XX. C.	Direct without calculation								
	XX. D.	Annual increment (balance of loan)								
II. A.	*Rate of accumulation varied. Rate of income unaltered.*									
	XIX. A.	Deductive	Original	$3\frac{1}{2}$	$3\frac{1}{2}$	1027·882	A	$3\frac{1}{2}$	3	1060·474
	XXII. C.	Annual increment (ratio)	Original	$3\frac{1}{2}$	$3\frac{1}{2}$	1027·882	A	$3\frac{1}{2}$	3	1060·474
	XXII. D.	Annual increment (ratio)	B	3	3	1060·474	C	3	$2\frac{1}{2}$	1093·909
II. B.	*Rate of accumulation varied. Rate of income varied.*									
	—	Not calculated	A	$3\frac{1}{2}$	3	1060·474	C	3	$2\frac{1}{2}$	1093·909
	—	Not calculated	Original	$3\frac{1}{2}$	$3\frac{1}{2}$	1027·882	B	3	3	1060·474
	XXII. E.	Annual increment (ratio)	Original	$3\frac{1}{2}$	$3\frac{1}{2}$	1027·882	C	3	$2\frac{1}{2}$	1093·909

CLASS II (A). *In which the rate of accumulation is varied, but in which the rate of income upon the present investments is unaltered.*

The following examples in Statement XXII. B. fall under this head:—

(1) In Variation A, as compared with the original conditions, the rate of income is in each case $3\frac{1}{2}$ per cent., but the rate of accumulation is reduced from $3\frac{1}{2}$ to 3 per cent. The effect as ascertained by the deductive method (Statement XIX. A.) is to increase the annual increment from £1027·882 to £1060·474. See also Calculation XXII. C., where the same result is obtained by the annual increment (ratio) method.

(2) In Variation C, as compared with Variation B, the rate of income is in each case 3 per cent., but the rate of accumulation in Variation C is reduced from 3 to $2\frac{1}{2}$ per cent. The effect, as will be seen from Statement XXII. B., is to increase the annual increment from £1060·474 to £1093·909. This adjustment is worked out in detail in Calculation XXII. D.

Chapter XIX deals very fully with the process of finding, by the deductive method, the amended annual instalment consequent upon a variation in the rate of accumulation only, taking as an example the original conditions as modified by Variation A. In both cases the present annual increment consists of:—

Income from investments	£347·648
Original annual instalment	680·234
	£1027·882

but under the original conditions this annual increment accumulated at $3\frac{1}{2}$ per cent., whereas in Variation A the rate of accumulation was reduced to 3 per cent. This requires an additional annual instalment of £32·592 to be set aside out of revenue or rate, as found by Calculation (XIX) 3. But it is apparent that this represents the deficiency in the accumulation of £1027·882 per annum at 3 per cent. instead of at $3\frac{1}{2}$ per cent., and is measured, not by the actual ratio between 3 and $3\frac{1}{2}$ per cent., but by the ratio existing between the respective amounts of £1 per annum for 13 years at those rates. These amounts

are given in Table III in the published tables of compound interest.

The same principle applies to Variation C, as compared with Variation B, as will be seen by the following Calculation XXII. D.

The remarks upon Calculation XXII. C. relating to Variation A, as compared with the original conditions, apply equally to this case.

Class II (B). *In which the rate of accumulation and the rate of income upon the present investments are both varied.*

The following examples in Statement XXII. B. fall under this head:—

(1) In Variation C, as compared with the original conditions, the rate of income is reduced from $3\frac{1}{2}$ to 3 per cent., and the rate of accumulation is reduced from $3\frac{1}{2}$ to $2\frac{1}{2}$ per cent. The effect, as found by Calculation XXII. E., is to increase the annual increment from £1027·882 to £1093·909.

(2) In Variation C, as compared with Variation A, the rate of income is reduced from $3\frac{1}{2}$ to 3 per cent., and the rate of accumulation is reduced from 3 to $2\frac{1}{2}$ per cent. The effect, as will be seen from Statement XXII. B., is to increase the annual increment from £1060·474 to £1093·909. This calculation is not worked out in detail, but follows from the premises as a matter of course.

(3) In Variation B, as compared with the original conditions, the rate of income is reduced from $3\frac{1}{2}$ to 3 per cent., and the rate of accumulation is reduced from $3\frac{1}{2}$ to 3 per cent. The effect, as will be seen from Statement XXII. B., is to increase the annual increment from £1027·882 to £1060·474. This calculation, also, is not worked out in detail.

In Calculation (XXII) E the original conditions are compared with Variation C in which there is a reduction in the rate of income from present investments from $3\frac{1}{2}$ to 3 per cent., or £49·664 per annum; and at the same time a reduction in the accumulation rate from $3\frac{1}{2}$ to $2\frac{1}{2}$ per cent. Although it is not so obvious as in those cases where there is a variation in the rate of accumulation only, yet the same rule applies, as will be seen by the following considerations. The method is a

combination of those previously considered. It resembles Variation A in that the rate of accumulation is reduced, and it is therefore necessary to increase the original annual increment in proportion to the respective amounts of £1 per annum for 13 years at the past and future rates of accumulation, as in Variation A., Calculation XXII. C. It resembles Variation B only to the extent that the actual decrease in the annual increment due to the reduced annual income of £49·664 must be added to the fund as an additional annual instalment to be provided out of revenue or rate. But in Variation B there is not any reduction in the rate of accumulation, as in this case; and in Variation B, therefore, the annual loss of income on the present investments is the actual measure of the deficiency in the annual instalment. In this case there is a reduction in the future income from the present investments accompanied by, and acted upon by, a reduction in the rate of accumulation; consequently if the original annual instalment be increased by the loss of income only, as in Variation B, the fund will lose the accumulation on that sum due to the reduction in the rate of accumulation. It is clear, therefore, that the original annual instalment must be increased, not by the actual loss of income, of ½ per cent., or £49·664, which under the original conditions accumulated at 3½ per cent., but by a larger annual amount which, accumulated at 2½ per cent. only, will at the end of the repayment period amount to the same sum. This question has been very fully discussed in Chapter XXI, dealing with Variation C, which contains a useful Statement XXI. C., which may be consulted in this connection. Therefore, the actual rate per cent. of income from the present investments does not enter into the calculation of the annual increment, which, as shown in Statement XXII. E., is exactly similar in principle to Calculations XXII. C. and XXII. D.

On comparing the whole of the above Calculations XXII. C., D. and E., it will be seen from the formulæ at the heading of each that they all follow the same rule, although the conditions in each are different. It may appear superfluous to include them all, but they will be referred to again in order to illustrate the variations in (1) the period of repayment, in Chapter XXIV, and (2) the period of repayment accompanied by a variation in the rate per cent. of accumulation, in Chapter XXVI.

The Annual Increment (balance of loan) Method. Three adjustments have been made relating to variations in the rate per cent. of accumulation as follows:—

(Variation C, as compared with the original conditions) which requires an original annual increment of	£1027·882
whereas in XXII. D. it is compared with Variation B., which requires an annual increment of	£1060·474
a difference of	£32·592

which is the increased annual instalment required in Variation A, as compared with the original conditions in consequence of the reduction in the rate of accumulation from $3\frac{1}{2}$ to 3 per cent. in Variation A. From the above data, as shown by Calculation XXII. C. and the deductive method previously described in Chapter XIX relating to Variation A, it is possible to deduce a further method of finding the future or amended annual instalment consequent upon a variation in the rate of accumulation accompanied or not by a change in the rate of income to be received upon the present investments representing the fund. This has been called the annual increment (balance of loan) method, and the rule may be stated as follows:—

(1) From the amount of loan repayable at the end of the original period of repayment (25 years)	£26495·00
(2) *deduct* the value of the present investments representing the fund at the time the adjustment is required to be made, namely, at the end of the 12th year	£9932·74
(3) and treat the balance of loan, viz.	£16562·26

as an original amount to be provided at the end of the unexpired portion (13 years) of the original repayment period by means of an annual increment based upon the future rate of accumulation.

(4) This, as shown by Calculation (XIX) 5, requires an annual increment to be accumulated at 3 per cent., of	£1060·474

(5) From this annual sum deduct the future annual income to be received from the present investments at the future rate per cent., whether unaltered, increased or reduced (in this case)	£347·648
(6) and the remainder	£712·826

is the future or amended annual sinking fund instalment to be set aside out of revenue or rate for the unexpired portion of the original repayment period, as ascertained in XIX. A. and XXII. C.

In a later chapter it will be found that the above rule, with modifications in the wording only, may be applied equally to variations in the period of repayment accompanied or not by variations in the rates per cent. of income and accumulation.

This will be shown in Chapter XXIV dealing with a variation in the period of repayment only, and in Chapter XXVI, dealing with a concurrent variation in the period of repayment and the rate of accumulation. For this reason the summary of the method at the head of this chapter has been so worded that it will apply to the whole of the problems above referred to.

The Rate per cent. **Calculation XXII. C.**

The Annual Increment (ratio) Method.

Class II. A. To find the amended annual increment (and therefrom the additional annual instalment) in a sinking fund in which the rate of accumulation is reduced, but in which the income from the present investments, and the period of repayment, remain unaltered.

The original conditions compared with Variation A by the deductive method. Statement XIX. A.

The rule relating to this method is stated at the head of Chapter XXIII.

Required the annual increment to be accumulated for a period of 13 years at 3 per cent., which is equivalent to an annual increment of £1027·882, to be accumulated for the same period at $3\frac{1}{2}$ per cent.

Income from investments, $3\frac{1}{2}$ per cent.

$$1027{\cdot}882 \left\{\frac{\text{Amount of £1 per annum, 13 years, } 3\frac{1}{2}\%}{\text{Amount of £1 per annum, 13 years, } 3\%}\right\} = 1060{\cdot}474$$

or by Table III, giving the amounts of £1 per annum :—

$$\frac{1027{\cdot}882 \times 16{\cdot}11303}{15{\cdot}6178} = 1060{\cdot}474$$

Log. Present annual increment ...	1027·882	3·0119434
add Log. Amount of £1 per annum Table III, 13 years, $3\frac{1}{2}$ per cent.	16·11303	1·2071771
	16562·26	4·2191205
deduct Log. Amount of £1 per annum Table III, 13 years, 3 per cent.	15·6178	1·1936196
Log. Amended annual increment		3·0255009

Amended annual increment		1060·474
To find the amended annual instalment:—		
deduct the income from investments, $3\frac{1}{2}$ per cent.		347·648
Amended annual instalment		712·826
being Original annual instalment ...	680·234	
Additional annual instalment	32·592	
		712·826

The Rate per cent. **Calculation XXII. D.**

The Annual Increment (ratio) Method.

Class II. A. To find the amended annual increment (and therefrom the additional annual instalment) in a sinking fund in which the rate of accumulation is reduced, but in which the income from the present investments, and the period of repayment, remain unaltered.

Variation B compared with Variation C.

This calculation is exactly similar in principle to XXII. C.

The rule relating to this method is stated at the head of Chapter XXIII.

Required the annual increment to be accumulated for a period of 13 years at $2\frac{1}{2}$ per cent., which is equivalent to an annual increment of £1060·474, to be accumulated for the same period at 3 per cent.

Income from investments, 3 per cent.

$$1060{\cdot}474 \left\{\frac{\text{Amount of £1 per annum, 13 years, 3\%}}{\text{Amount of £1 per annum, 13 years, } 2\frac{1}{2}\%}\right\} = 1093{\cdot}909$$

or by Table III, giving the amounts of £1 per annum:—

$$\frac{1060{\cdot}474 \times 15{\cdot}6178}{15{\cdot}14044} = 1093{\cdot}909$$

Log. Present annual increment ...	1060·474	3·0255009
add Log. Amount of £1 per annum Table III, 13 years, 3 per cent.	15·6178	1·1936196
	16562·26	4·2191205
deduct Log. Amount of £1 per annum Table III, 13 years, $2\frac{1}{2}$ per cent.	15·14044	1·1801386
Log. Amended annual increment		3·0389819
Amended annual increment		1093·909

To find the amended annual instalment:—

deduct the income from investments, 3 per cent.		297·984
Amended annual instalment		795·925
being Original annual instalment ...	762·490	
Additional annual instalment	33·435	
		795·925

The Rate per cent. **Calculation XXII. E.**

The Annual Increment (ratio) Method.

Class II. B. To find the amended annual increment (and therefrom the additional annual instalment) in a sinking fund in which the rate of accumulation and the income from the present investments are both reduced, but in which the period of repayment remains unaltered.

The original conditions compared with Variation C.

The rule relating to this method is stated at the head of Chapter XXIII.

Required the annual increment to be accumulated for a period of 13 years at $2\frac{1}{2}$ per cent., which is equivalent to an annual increment of £1027·882, to be accumulated for the same period at $3\frac{1}{2}$ per cent.

The rate of income from investments is reduced from $3\frac{1}{2}$ to 3 per cent.

$$1027{\cdot}882 \left\{\frac{\text{Amount of £1 per annum, 13 years, } 3\frac{1}{2}\%}{\text{Amount of £1 per annum, 13 years, } 2\frac{1}{2}\%}\right\} = 1093{\cdot}909$$

or by Table III, giving the amounts of £1 per annum:—

$$\frac{1027{\cdot}882 \times 16{\cdot}11303}{15{\cdot}14044} = 1093{\cdot}909$$

Log. Present annual increment ...	1027·882	3·0119434
add Log. Amount of £1 per annum Table III, 13 years, $3\frac{1}{2}$ per cent.	16·11303	1·2071771
	16562·26	4·2191205
deduct Log. Amount of £1 per annum Table III, 13 years, $2\frac{1}{2}$ per cent.	15·14044	1·1801386
Log. Amended annual increment		3·0389819
Amended annual increment		1093·909
To find the amended annual instalment:—		
deduct the income from investments, 3 per cent.		297·984
Amended annual instalment		795·925
being Original annual instalment ...	680·234	
Additional annual instalment	115·691	
		795·925

CHAPTER XXIII.

SINKING FUND PROBLEMS, RELATING TO THE RATE PER CENT. OF ACCUMULATION (*Continued*).

DERIVATION OF A RULE AND FORMULA RELATING TO A VARIATION IN THE RATE PER CENT. OF ACCUMULATION BASED UPON THE FOREGOING RESULTS BY THE ANNUAL INCREMENT (RATIO) METHOD.

The Annual Increment (ratio) Method.

The rule as to a variation in the rate of accumulation may be stated as follows, using the terms as explained at the head of Chapter XXII. *Statement XXII. C.*

RULE. *To find the amended annual instalment to be set aside, and added to the existing sinking fund,*

to be accumulated in future at a rate per cent. greater or less than the rate at which the present annual instalment was calculated

(the future rate),

and to be set aside during the unexpired portion of the original repayment period

(the unexpired period).

Proceed as follows:—

(1) *Ascertain the present annual increment of the fund, as described in Chapter XXII.*

(2) *Multiply the annual increment so found by the amount of £1 per annum at the past rate for the unexpired period.*

(3) *Divide the above product by the amount of £1 per annum at the future rate for the same unexpired period.*

(4) *The amount so found will represent the future or amended annual increment of the fund under the new conditions. The amended annual sinking fund instalment may be found by deducting therefrom the future annual income from the present investments representing the fund.*

(5) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XIX. B.*

(6) *Prepare the usual pro forma account previously recommended.* *Pro forma Account No. 7.*

This rule will not apply to cases in which the rate of income on investments only is varied. Such problems may be solved by the simple direct method, without calculation, described in Chapter XX, Statement C. It is imperative, in using this method, that the future rate of accumulation and the rate of income upon the present investments shall be uniform during the whole of the unexpired portion of the period of repayment.

THE ANNUAL INCREMENT (RATIO) METHOD. *Derivation of a rule and formula relating to a variation in the rate per cent. of accumulation.* The previous chapters illustrate the various methods of adjusting the annual sinking fund instalment in consequence of all possible combinations of changes in the rates per cent. of income and accumulation, with the result that the variations have been divided into two broad groups depending upon the future rate of accumulation, as shown in Statement XXII. B. All variations relating to the rate of income and accumulation may be adjusted by making the calculation by the deductive method described in Chapter XIX, but where the variation affects only the rate of income upon investments, and the rate of accumulation remains unaltered, the deductive method is superfluous and may be replaced by the more simple direct method without calculation, as described in Chapter XX, Statement XX. C. The variations affecting the rate of accumulation have been divided into two sub-classes according as the variation in the rate of accumulation is accompanied or not by a change in the rate of income upon the present investments.

The effect of a variation in the rate of income upon the present investments has been eliminated by ascertaining the actual amount of such income to be yielded annually in future, and treating the same as an anunuity to be paid into the fund and accumulated along with the amended annual instalment. These two annual sums have been combined under the term annual increment which is acted upon by the rate of accumulation only, and the enquiry is therefore confined to the rate of accumulation.

By this method the original annual instalment, as such, takes only a minor place in the calculation which is made in terms of the annual increment. Having found the future or amended annual increment required, under the new conditions, to be paid into the fund and accumulated for the unexpired portion of the original repayment period, the future annual income from investments is deducted therefrom in order to ascertain the future or amended annual instalment to be set aside out of revenue or rate. The difference between this amended instalment and the original instalment is the additional annual charge to revenue or rate due to the variation in the rates of both income and accumulation.

Having reduced all problems to terms of the present annual increment at the date of making the adjustment, it is found that this annual sum must be increased or reduced in a definite ratio depending upon the original and amended rates of accumulation. If it be required to ascertain the respective amounts of principal which will provide a given annual sum in perpetuity at two varying rates per cent., they will be inversely proportional to the respective rates. But if it be required to find, as in the problems now under discussion the respective annuities which will amount to a given sum at the end of a given term at varying rates per cent., the element of accumulation enters into the calculation, although the resulting annuities are still, in a sense, in inverse ratio to the rates per cent. Very little consideration will show that the ratio, instead of being expressed in terms of the actual rates per cent., must be expressed in terms of the amounts of £1 per annum at the respective rates per cent., both for a number of years equal to the unexpired portion of the period of repayment. This latter provision is important; it is not the factor (R) so often used (which is £1 increased by interest for one year) but $\frac{R^N - 1}{r}$, in which N represents the number of years in the unexpired portion of the repayment period, and which expresses the amount of an annuity of £1 in any number of years, as shown in Chapter VI, dealing with Table III. In the previous discussion of the subject in Chapter XXII this method has been applied to three of the examples previously considered, and results have been obtained identical with those found by the deductive method. These results are shown in Calculations XXII. C., D., and E. On referring to these calculations it will be seen that in each case the actual working is prefaced by a formula commencing with the present annual increment

at the time the adjustment is required to be made, which annual increment is multiplied by a fraction. In all cases the numerator of this fraction is the amount of £1 per annum at the past rate of accumulation governing the above annual increment up to the time of making the adjustment. The denominator of the fraction is, in each case, the amount of £1 per annum at the future or substituted rate of accumulation which will govern the future or amended annual increment required.

The following table will make the matter clear and will be useful for future reference when considering the question of a variation in the rate per cent. of accumulation accompanied by a variation in the period of repayment. It shows the respective variations in the rate of accumulation in the examples previously used to illustrate the derivation of a rule and formula applying to all such variations, namely, the annual increment (ratio) method:—

	Rate of Accumulation reduced:—		Amount of £1 per annum, for 13 years.	
Calculation.	From	To	Numerator.	Denominator.
XXII. C.	$3\frac{1}{2}$ per cent.	3 per cent.	$3\frac{1}{2}$ per cent.	3 per cent.
XXII. D.	3 ,,	$2\frac{1}{2}$,,	3 ,,	$2\frac{1}{2}$,,
XXII. E.	$3\frac{1}{2}$,,	$2\frac{1}{2}$,,	$3\frac{1}{2}$,,	$2\frac{1}{2}$,,

In the whole of the progressive examples used to illustrate the consideration of the general question of variations in the rates per cent. of income upon investments and of accumulation, a gradual reduction in both rates has been assumed. It has been frequently pointed out that the methods already adopted will apply equally to an increase in such rates, and an inspection of Statement XXII. B. will confirm this. It will be seen later, in Chapter XXVI, when considering the question of a variation in the rate of accumulation, complicated by a variation in the period of repayment, that the same rule holds good, seeing that the numerator of the fraction is always based upon the past rate of accumulation, and the denominator upon the future rate.

A rule and formula may now be stated, based upon the foregoing considerations and upon Calculations XXII. C., D., and E., for finding by direct calculation from the present annual increment (not the annual sinking fund instalment) the future or amended annual increment due to a variation in the rate of accumulation, whether accompanied or not by a variation in the rate of income upon the investments representing the

fund at the time of making the adjustment. In stating the rule and formula relating to a variation in the rate of accumulation in this chapter, as well as the rules relating to a variation in the period of repayment in Chapter XXV, and a concurrent variation in both period and rate of accumulation in Chapter XXVI, the abbreviated terms which are given at the head of Chapter XXII will be used, as follows:—

The Past Rate denotes the rate of accumulation upon which was based the original annual instalment included in the present annual increment.

The Future Rate denotes the rate of accumulation to be used instead of the past rate, to calculate the amended annual increment. It will be the same as the past rate in problems involving a variation in the period of repayment only without any variation in the rate of accumulation.

The Unexpired Period denotes the unexpired portion at the time of making the adjustment of the original repayment period upon which the present or original annual instalment was based.

The Substituted Period denotes the increased or reduced number of years over which the future or amended annual instalment shall be spread, and at the end of which the full amount of the loan will be repayable. It will be the same as the unexpired period in problems involving a variation in the rate of accumulation only, without any variation in the period of repayment.

The rule as to a variation in the rate of accumulation only (the annual increment (ratio) method) is stated in full at the head of this chapter.

The above rule is sufficiently explicit, but as it will, in Chapter XXVI, be combined with the rule relating to a variation in the period of repayment, it is expressed as a formula as follows:—

VARIATION IN THE RATE OF ACCUMULATION.

The Annual Increment (ratio) Method.

$$\left\{\begin{array}{c}\text{Present}\\\text{annual}\\\text{increment.}\end{array}\right\} \times \left\{\frac{\begin{array}{c}\text{Amount of £1 per annum}\\\text{at past rate}\\\text{for unexpired period.}\end{array}}{\begin{array}{c}\textit{Amount of £1 per annum}\\\textit{at future rate}\\\textit{for unexpired period.}\end{array}}\right\} = \left\{\begin{array}{c}\text{Future}\\\text{or}\\\text{amended}\\\text{annual}\\\text{increment.}\end{array}\right\}$$

The amounts of £1 per annum in the above rule and formula are at varying rates per cent. of accumulation, but are for the same number of years.

Calculation XXII. C. will now be expressed in terms of the above formula, but in this case the problem will be inverted to apply to an increase in the rate of accumulation instead of a decrease, as follows:—

$$1060{\cdot}474 \times \left(\frac{15{\cdot}61779}{16{\cdot}11303}\right) = 1027{\cdot}882.$$

In Chapter XXVI this calculation will be combined with the similar calculation shown in Chapter XXV, but relating to a variation in the period of repayment.

It will be noticed that the above rule, and the formula expressing it, do not contain any reference to the future rate of income to be yielded by the present investments representing the fund, and that the sole governing factor is the varying rate of accumulation. This rule and formula will apply equally to an increase or decrease in the future rate of accumulation, and it is important to remember that an increase in the rate of accumulation will cause a reduction in the annual instalment to be charged to revenue or rate account in future years; an increase in the repayment period will, on the other hand, involve a decrease in the future annual instalment.

The object of expressing the above rule in formula form will be seen later in Chapter XXV, when discussing the adjustment of the annual instalment in consequence of a variation in the period of repayment only, and also when discussing, in Chapter XXVI, the adjustment in the annual instalment due to a variation in the period of repayment accompanied by a variation in the rate of accumulation.

In Chapter XXVI both the above formulæ will be combined, but in this case Calculation XXII. C. will be used in an inverted form in order to obtain an example of an increase in the rate of accumulation from 3 to $3\frac{1}{2}$ per cent. which will be used as the basis of Calculation XXVI. C.

On comparing the above formula with the formula in Chapter XXV, relating to a variation in the period of repayment, it will be noticed that the denominator in the above formula is the same as the numerator in the formula in Chapter XXV.

CHAPTER XXIV.

SINKING FUND PROBLEMS, RELATING TO THE REDEMPTION PERIOD.

A VARIATION IN THE PERIOD OF REPAYMENT WITH OR WITHOUT ANY VARIATION IN THE RATES PER CENT. OF INCOME OR ACCUMULATION. SUMMARY OF METHODS. GENERAL CONSIDERATIONS AS TO THE REDEMPTION PERIOD. THE DEDUCTIVE METHOD. THE ANNUAL INCREMENT (RATIO) METHOD, AND THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN BY THE OPERATION OF THE AMENDED ANNUAL INSTALMENT.

Summary of the methods of adjustment.

(I) The deductive method, as summarised below (see note).
Statement XXIV. A.

(II) The direct method, without calculation, as summarised at the head of Chapter XX, will not apply to this variation.

(III) The annual increment (balance of loan) method, as summarised at the head of Chapter XXII.
Statement XXIV. D.

(IV) The annual increment (ratio) method, as summarised at the head of Chapter XXV. *Statement XXIV. C.*

Note. The terms used in the following summary are fully explained at the head of Chapter XXII. The deductive method summarised below relates only to a variation in the period of repayment, and is of limited application, in that the rates of accumulation and of income from investments are both the same and remain unaltered. The method described in Chapter XIX is more generally applicable, and should be followed in all cases.

SUMMARY OF THE DEDUCTIVE METHOD, *of ascertaining the amended annual sinking fund instalment due to a variation in the period of repayment only, without any variation in the rates per cent. of accumulation or of income from the present investments representing the fund, both of which must be the same.* *Statement XXIV. A.*

(1) *Ascertain the value of the present investments as previously described.*

(2) *Calculate the amount thereof, if accumulated for the substituted repayment period at the past unaltered rate of accumulation.* *Calculation (XXIV) 1.*

(3) *Calculate the amount of an annuity equal to the present or original annual instalment for the substituted period at the past unaltered rate of accumulation.*
Calculation (XXIV) 2.

(4) *The amount found in (2) added to the amount found in (3) will represent the amount of original loan which will be provided thereby at the end of the substituted period of repayment.*

(5) *Deduct the sum found in (4) from the amount of original loan, and the remainder represents the portion of original loan which will be unprovided for by the accumulation of the present investments and the present or original annual instalment at the past unaltered rate of accumulation.*

(6) *Calculate the additional annual sinking fund instalment which at the past unaltered rate of accumulation will amount to the balance of loan found in (5) at the end of the substituted period of repayment.*
Calculation (XXIV) 3.

(7) *The additional annual instalment found in (6) added to the original or present annual instalment, as in (3), gives the future amended annual instalment to be set aside and added to the fund during the substituted period of repayment.*

(8) *Prepare a statement showing the final repayment of the loan by the operation of the fund under the amended conditions.* *Statement XXIV. B.*

(9) *Prepare a pro forma account showing the amount which should be in the fund at the end of each year of the substituted repayment period.*
Pro forma Account, No. 10.

Memo. The above method will apply equally to an increase or reduction in the period of repayment.

General Considerations. It very rarely happens that there is any alteration in the period originally allowed for the repayment of any individual loan of a local authority. It may be taken as a general rule that in the special or general Act, provisional order, or sanction of the Local Government Board, authorising the expenditure and the consequent borrowing, there is a specified period imposed for the final repayment of the loan out of revenue or rate, and this period is strictly adhered to. The Local Government Board have power under the Local Government Act, 1888, and the Public Health Acts Amendment Act, 1890, to extend or vary the periods within which loans may be discharged, but this power is limited to the consolidation of debt, and the exercise of such power is therefore confined to the equation of the repayment periods of the several loans so consolidated. The discussion of this part of the subject will be deferred to Chapter XXXII, where it will be fully considered. It is different with the sinking funds set aside to repay the loan debt of commercial or financial undertakings. In these cases the conditions are much more elastic than in the case of local authorities, and almost every kind of variation is met with in practice. These problems may arise at the time the sinking fund is inaugurated in order to meet any special obligations imposed at the time the loan is arranged, or to meet any future contingency, which it is anticipated may arise during the continuation of the fund. It may also happen that events occur after the fund has been in operation for some years which require that the period of repayment shall be increased or reduced, and any alteration in the period may be, and generally is, accompanied by a variation in the rate per cent. of accumulation. Any variation in the rate of interest payable to the loan holders rarely affects the sinking fund instalment, and may generally be ignored, but in all questions of this nature it is most important to ascertain the whole of the conditions in order that the proper adjustment may be made.

The Methods of Adjustment. *The deductive method.* Although a shorter method has been found of making the adjustment in the annual instalment, in the present instance the deductive method will again be first used, afterwards making the same adjustment by the methods described as the annual increment (ratio) method and the annual increment (balance of loan) method.

In this chapter the variation will be assumed to relate only to the period of repayment without any complication arising in consequence of a variation in the rate of accumulation or of income upon the present investments. In the following chapter (XXV) the annual increment (ratio) method will be reduced to a rule and formula relating to the period of repayment only, in a similar manner to that adopted in Chapter XXIII, relating to the rate of accumulation. It will, however, sometimes happen that an adjustment is required to be made owing to a concurrent variation in the rate of income to be received from the present investments and also from the investment of the future accretions to the fund, and these again may be at different rates. All questions arising out of a variation in the rate per cent. generally, have been considered in previous chapters, and the adjustment due to a simultaneous variation in both period and rate per cent. will be deferred to Chapter XXVI.

The present problem will be illustrated by the now familiar example of the sinking fund already discussed, which relates to the repayment of a loan of £26,495 at the end of 25 years, requiring an annual instalment of £680·234, to be set aside and accumulated at $3\frac{1}{2}$ per cent., as found by Calculation (XV) 1.

Circumstances have arisen which impose upon the undertaking the necessity to accelerate the final redemption of the loan indebtedness by the operation of the fund. It is not necessary to enquire into the special reason for such acceleration because the principle is the same in any event. The adjustment will again be based upon the position of the fund at the end of the 12th year. The undertaking or company was originally required to repay the loan of £26,495 at the end of the 25th year, namely, in 13 years from the present time, and, towards this, there is in the fund the proper calculated amount, which is represented by investments valued at £9932·74, as found by Calculation (XV) 2, yielding an assured future annual income, at $3\frac{1}{2}$ per cent., of £347·648. The altered conditions demand that the operation of the fund shall be accelerated and that the original annual instalment shall be increased to such an amount as will repay the loan in 8 years from the present time instead of at the end of 13 years thereby reducing the original repayment period from 25 to 20 years.

This reduction in the period affects the future accumulation of the annual instalment of £680·234, as originally calculated. and also the future accumulation of the amount of £9932·74 now in the fund. In order to compare the resulting increased

annual instalment with the original instalment it will be assumed that the original estimated rate of accumulation, namely, 3½ per cent., will continue to be received during the remaining 8 years, both as regards the income from the present investments and the amended annual instalment.

All the present factors will be again reduced to equivalent amounts of original loan which will be provided at the end of the substituted period of 8 years by the accumulation of such factors in order to ascertain, by the deductive method, as shown in Statement XXIV. A., the portion of original loan which remains to be provided by an additional annual instalment. If the rate of accumulation remains unaltered the reduction in the period of repayment will have the effect of increasing the annual instalments as originally calculated. If, on the contrary, the unexpired period of 13 years be extended instead of reduced, there will be an apparent surplus in the fund which will lead to a reduction in the annual instalment.

The additional annual instalment required, as shown in Statement XXIV. A., by the deductive method, is £801·862. The balance of loan, £7258·21, shown in Statement XXIV. A., which will be unprovided for owing to the reduction in the redemption period from 13 years to 8 years is made up as follows:—

Present investments	£9932·74	
Amount thereof, accumulated for 13 years at 3½ per cent. Calculation (XVII) 2	£15534·38	
Amount thereof, accumulated for 8 years at 3½ per cent. Calculation (XXIV) 1	£13079·53	
		£2454·85
Original annual instalment:—		
Amount of £680·234 per annum, accumulated for 13 years at 3½ per cent. Calculation (XV) 5	£10960·62	
Amount of £680·234 per annum, accumulated for 8 years at 3½ per cent. Calculation (XXIV) 2	£6157·26	
		£4803·36
Balance of loan unprovided for		£7258·21

It has thus been ascertained that the ultimate amount of loan which will be unprovided at the end of the substituted period in consequence of the reduction in the original redemption period is £7258·21, and this deficiency has been divided between the accumulations of the present investments and of the original annual instalment. The portion of the deficiency due to the reduced accumulation of the present investments is £2454·85, and has been expressed in terms of the capital value, but it may also be expressed in terms of the annual income of £347·648 to arise from the present investments, as follows:—

Amount of £347·648 per annum in 13 years at $3\frac{1}{2}$ per cent.	Calculation (XXIV) 4	£5601·66
Amount of £347·648 per annum in 8 years at $3\frac{1}{2}$ per cent.	Calculation (XXIV) 5	£3146·81
		£2454·85

Statement XXIV. A. shows that the reduction in the period of repayment from 25 years to 20 years (but with the same rate of accumulation) taking place at the end of the 12th year, results in an increased annual burden of £801·862 chargeable against the revenue of the undertaking. It only now remains to review the operation of the fund under the altered conditions in order to ascertain that the amended annual instalment of £1482·096 so found will carry out the purpose of the fund, namely, to repay the loan of £26,495, but at the end of 20 instead of 25 years. This is shown in Statement XXIV. B., and by the pro forma account, No. 10.

THE ANNUAL INCREMENT (RATIO) METHOD. In previous chapters dealing with each of the variations in the rates per cent. of income and accumulation, the additional annual instalment was first ascertained by the deductive method, as fully described in Chapter XIX. This method is based essentially upon the ultimate separate accumulation at the future rate of each of the present factors of the fund, namely, the annual instalment as originally calculated, the value of the present investments, and the future income to arise therefrom, all of which were reduced to equivalent amounts of original loan which they will individually provide at the end of the period of redemption. In Chapter XXII the whole of these adjustments were again made by direct calculations based upon

the annual increment of the fund as defined in Chapter XIV, and it was found that by this means it was possible to simplify the calculation and eliminate altogether the effect of any variation in the rate of income to be received in future upon the present investments representing the fund. It was found that there is an exact ratio existing between the present and future annual increments depending upon the respective amounts of £1 per annum; and in Chapter XXIII, relating solely to the rate of accumulation, this method of calculation was reduced to a rule and formula, called the annual increment (ratio) method.

For the purpose of the following adjustment the present annual increment, which is the basis of the calculation, is made up as follows:—

Original annual instalment	£680·234
Income from present investments	347·648
	£1027·882

The above annual income from the present investments, as in all adjustments made by this method, is the amount which has been received in the past, and is not the amount which will be yielded thereby during the substituted period of repayment.

This is one of the fundamental principles of the annual increment (ratio) method, as fully explained in the opening paragraphs of Chapter XXII. This method will now be applied to a variation in the period of repayment, as shown in Calculation XXIV. C.

The Annual Increment (balance of loan) Method. It has been found in Chapter XXII, dealing with a variation in the rate of accumulation, that the future or amended annual increment, and therefrom the future or amended annual instalment, may be obtained by deducting the value of the present investments representing the fund, from the total amount of loan repayable at the end of the prescribed period, and treating the balance as an original amount to be provided by an annual sum to be accumulated during the unexpired portion of the original repayment period at the future amended rate of accumulation. The annual sum so found is the equivalent of the future or amended annual increment, and the future or amended annual instalment under the new conditions

is found by deducting therefrom the annual income to be received in future upon the present investments representing the fund, at any rate per cent. whether increased or reduced. This is the annual increment (balance of loan) method, and although its derivation is not described until Chapter XXII, it has been used in previous chapters. Statement XXIV. D. following gives details of the present example worked out by this method.

The Redemption Period. Statement XXIV. A.

The Deductive Method.

Showing the method of adjusting the annual instalment in consequence of a variation in the redemption period without any variation in the rate per cent. of accumulation or of income from the present investments, both of which rates are the same.

If these rates are unequal or are varied proceed as in Chapter XIX. A.

Conditions before adjustment, at end of 12th year:

Amount of original loan, repayable in 25 years	£26,495
Amount in the fund, at end of 12th year ...	£9932·74
Present annual income (previously) received therefrom, at $3\frac{1}{2}$ per cent., per annum ...	£347·648
Present annual instalment, to be accumulated for 13 years, at $3\frac{1}{2}$ per cent.	£680·234
Present annual increment	£1027·882

Variation from the above conditions :—

The period during which the loan shall be redeemed is reduced from 13 to 8 years.

The substituted period of repayment 8 years.

		Equivalent amount of original loan.
Present investments (at end of 12th year)	£9932·74	
Amount thereof, accumulated for 8 years at $3\frac{1}{2}$ per cent. Calculation (XXIV) 1		£13079·53
Original annual instalment	£680·234	
Amount of £680·234 per annum, for 8 years at $3\frac{1}{2}$ per cent. Calculation (XXIV) 2		£6157·26
Provision already made, will repay loan of		£19236·79

Additional annual instalment required :—

Balance, being amount of original loan unprovided for, owing to the above decrease in the redemption period requiring an additional annual instalment to be set aside and accumulated for 8 years at $3\frac{1}{2}$ per cent. ... £7258·21

Additional annual instalment

Calculation (XXIV) 3 £801·862

Amount of original loan ... ... £26495·00

Amended annual increment :—

Annual income from investments...	£347·648
Amended annual instalment	£1482·096
	£1829·744

The Redemption Period. Statement XXIV. B.

Showing the final repayment of the loan, by the operation of the sinking fund after making the adjustment in the annual instalment consequent upon a reduction in the period of repayment, without any variation in the rate per cent. of accumulation, or of income from the present investments.

		Equivalent amount of original loan.
Present investments (at end of 12th year)		£9932·74
Amended annual increment :—		
Original annual instalment	£680·234	
Additional annual instalment ...	801·862	
Total out of revenue	£1482·096	
Income from investments	347·648	
Total	£1829·744	
Amount thereof, accumulated for 8 years at $3\frac{1}{2}$ per cent. Calculation (XXIV) 6		£16562·26
Amount of original loan		£26495·00
Amended annual instalment	£1482·096	

The Redemption Period. Calculation XXIV. C.

The Annual Increment (ratio) Method.

To find the amended annual increment (and therefrom the amended annual instalment) in a sinking fund in which the original period of repayment is varied, accompanied or not by any variation in the rates of accumulation or of income from the present investments.

The rule relating to this method is stated at the head of Chapter XXV.

Required the annual increment to be accumulated for a period of 8 years, which is equivalent to an annual increment of £1027·882, to be accumulated for a period of 13 years, the rate of accumulation in both cases being $3\frac{1}{2}$ per cent.

$$1027{\cdot}882\left\{\frac{\text{Amount of £1 per annum, 13 years, } 3\tfrac{1}{2}\%}{\text{Amount of £1 per annum, 8 years, } 3\tfrac{1}{2}\%}\right\}=1829{\cdot}744$$

or by Table III, giving the amounts of £1 per annum.

$$\frac{1027{\cdot}882\times 16{\cdot}11303}{9{\cdot}05168}=1829{\cdot}744$$

Log. Present annual increment ...	1027·882	3·0119434
add Log. Amount of £1 per annum Table III, 13 years, $3\frac{1}{2}$ per cent.	16·11303	1·2071771
	16562·26	4·2191205
deduct Log. Amount of £1 per annum Table III, 8 years, $3\frac{1}{2}$ per cent.	9·05168	0·9567296
Log. Amended annual increment		3·2623909
Amended annual increment		1829·744

To find the amended annual instalment:—

deduct the income from investments, $3\frac{1}{2}$ per cent.		347·648
Amended annual instalment		1482·096
being Present annual instalment ...	680·234	
Additional annual instalment	801·862	
		1482·096

The Redemption Period. Statement XXIV. D.

The Annual Increment (balance of loan) Method.

To find the amended annual sinking fund instalment consequent upon a variation in the period of repayment with or without any variation in the rate of income to be received from the present investments or in the rate of accumulation.

For Rule see Chapter XXII.

Amount of original loan (25 years)		£26495·00
deduct amount in the fund at the end of the 12th year		£9932·74
Balance of loan		£16562·26
Amended annual increment, to be added to the fund, and accumulated at $3\frac{1}{2}$ per cent., to provide this amount at the end of 8 years. Calculation XXIV. C.		£1829·744
deduct therefrom income to be received from the present investments, £9932·74, at $3\frac{1}{2}$ per cent.		£347·648
Amended annual instalment		£1482·096
being Original annual instalment ...	£680·234	
Additional annual instalment	801·862	
		£1482·096

Pro forma Sinking Fund Account, No. 10.

A Variation in the Redemption Period.

Loan of £26,495, repayable at the end of 25 years.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the increased annual instalment of £1482·096.

Statement XXIV. B. Rate of accumulation, $3\frac{1}{2}$ per cent.

<table>
<tr><th>Year.</th><th>Amount in the fund at beginning of year.</th><th>Income received from investments 3½ per cent.</th><th>Annual sinking fund instalment.</th><th>Amount in the fund at end of year.</th><th>Year</th></tr>
<tr><td>1</td><td></td><td></td><td></td><td></td><td>1</td></tr>
<tr><td>2</td><td></td><td></td><td></td><td></td><td>2</td></tr>
<tr><td>3</td><td></td><td></td><td></td><td></td><td>3</td></tr>
<tr><td>4</td><td colspan="4" rowspan="5">The amount in the fund at the end of the 12th year, £9932·744, is the correct calculated amount, as shown by Calculation (XV) 2, and by the pro forma account, No. 1, Chapter XV.</td><td>4</td></tr>
<tr><td>5</td><td>5</td></tr>
<tr><td>6</td><td>6</td></tr>
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<tr><td>11</td><td></td><td></td><td></td><td></td><td>11</td></tr>
<tr><td>12</td><td></td><td></td><td></td><td>9932·744</td><td>12</td></tr>
<tr><td>13</td><td>9932·744</td><td>347·648</td><td>1482·096</td><td>11762·488</td><td>13</td></tr>
<tr><td>14</td><td>11762·488</td><td>411·687</td><td>1482·096</td><td>13656·271</td><td>14</td></tr>
<tr><td>15</td><td>13656·271</td><td>477·969</td><td>1482·096</td><td>15616·336</td><td>15</td></tr>
<tr><td>16</td><td>15616·336</td><td>546·572</td><td>1482·096</td><td>17645·004</td><td>16</td></tr>
<tr><td>17</td><td>17645·004</td><td>617·575</td><td>1482·096</td><td>19744·675</td><td>17</td></tr>
<tr><td>18</td><td>19744·675</td><td>691·063</td><td>1482·096</td><td>21917·834</td><td>18</td></tr>
<tr><td>19</td><td>21917·834</td><td>767·124</td><td>1482·096</td><td>24167·054</td><td>19</td></tr>
<tr><td>20</td><td>24167·054</td><td>845·850</td><td>1482·096</td><td>26495·000</td><td>20</td></tr>
<tr><td>21</td><td></td><td></td><td></td><td></td><td>21</td></tr>
<tr><td>22</td><td></td><td></td><td></td><td></td><td>22</td></tr>
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CHAPTER XXV.

SINKING FUND PROBLEMS, RELATING TO THE REDEMPTION PERIOD (*Continued*).

DERIVATION OF A RULE AND FORMULA RELATING TO A VARIATION IN THE PERIOD OF REPAYMENT BASED UPON THE FOREGOING RESULTS BY THE ANNUAL INCREMENT (RATIO) METHOD.

The Annual Increment (ratio) Method.

The rule as to a variation in the period of repayment, may be stated as follows, using the terms explained at the head of Chapter XXII. *Statement XXIV. C.*

RULE. *To find the amended annual instalment to be set aside, and added to the existing sinking fund,*
to be accumulated in future at the same rate per cent. at which the present annual instalment was calculated
(the future rate),
and to be set aside for a reduced or increased number of years as compared with the unexpired portion of the original repayment period
(the substituted period).

Proceed as follows:—

(*1*) *Ascertain the present annual increment of the fund, as described in Chapter XXII.*

(*2*) *Multiply the annual increment so found by the amount of £1 per annum at the future rate for the unexpired period.*

(*3*) *Divide the above product by the amount of £1 per annum, at the future rate for the substituted period.*

(*4*) *The amount so found will represent the future or amended annual increment of the fund under the new conditions. The amended annual sinking fund instalment may be found by deducting therefrom the future annual income from the present investments representing the fund.*

(5) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XXIV. B.*

(6) *Prepare the usual pro forma account previously recommended.* *Pro forma Account, No. 10, Chapter XXIV.*

It is imperative, in using this method, that the future rate of accumulation and the rate of income from the present investments, shall be uniform during the whole of the substituted period of repayment.

THE ANNUAL INCREMENT (RATIO) METHOD. *Derivation of a rule and formula, relating to a variation in the period of repayment.* The subject of enquiry in Chapter XXIII is the derivation of a rule and formula by which to ascertain the future or amended annual increment, and therefrom the amended annual instalment, due to a variation in the rate of accumulation only. The present object is to find a similar rule and formula which will apply to a variation in the period of repayment, and the method to be adopted will be the same in principle. In discussing the effect of a variation in the rate of accumulation upon the future or amended annual increment in Chapter XIX, Variation A, the amended annual increment was ascertained by the somewhat roundabout, although instructive, deductive method there described (Statement XIX. A.). This method was used purposely in order to emphasise the principles involved and to show the effect of the variation in the rate of accumulation upon each of the actual factors of the fund, namely, the present investments, the annual income to arise therefrom, and the original annual sinking fund instalment. This deductive method of enquiry was again adopted in Chapter XX (Variation B, rate of income upon investments), and the amended annual increment was ascertained as shown in Statement XX. A. In Chapter XXI, the same method was applied to ascertain the amended annual increment due to a dual variation in the rates per cent. of accumulation and of income upon investments (Variation C), and the result is contained in Statement XXI. A.

Chapter XXII contains a tabular summary (XXII. A.) of the results obtained in all the above investigations into the effect of variations in the rate per cent. This summary shows that in each of the above cases the original and amended annual increments bear a certain definite ratio one to the other; and

from this ratio it is possible to derive a rule and formula by which to derive the amended annual instalment directly from the original annual instalment.

In the above examples the period of repayment remained unaltered, and it has been ascertained that any variation in the rate per cent. of accumulation has the effect of increasing or reducing the present annual increment in proportion to the ratio existing between the amounts of £1 per annum for the unexpired portion of the original repayment period at the past and future rates of accumulation respectively. A similar method will now be applied to the derivation of a rule and formula by which to find the future or amended annual increment, and therefrom the amended annual instalment, due to a variation in the period of repayment, the rate of accumulation remaining the same, and it will be demonstrated by means of the results obtained in the example just considered in Chapter XXIV. In this instance there is a present annual increment, receivable for 13 years, composed of:—

Original annual instalment...	£680·234
Income from present investments	347·648
	£1027·882

and this present annual increment, if accumulated at $3\frac{1}{2}$ per cent. for 13 years, is sufficient to provide a definite amount of loan at the end of that time. The above annual income from the present investments, as in all adjustments made by this method, is the amount which has been received in the past, and is not the amount of income which will be yielded thereby during the unexpired or substituted period of repayment. This is one of the fundamental principles of the annual increment (ratio) method, as fully explained in the opening paragraphs of Chapter XXII. For this purpose it is not necessary to know the actual amount of the loan, but the above present annual increment may be treated as a simple annuity certain, for a period of 13 years, to be accumulated at $3\frac{1}{2}$ per cent. It is required to ascertain the equivalent annuity or annual increment accumulating at the same rate to repay the same loan, but at the end of a term of 8 years instead of at the end of 13 years. It has already been ascertained, in Chapter XXIV, that this equivalent annual increment is £1829·744. In the case of the previous Calculations XXII. C., D., and E., the

period of repayment remained the same, but the rate of accumulation varied; consequently the ratio was expressed in terms of the amounts of £1 per annum at the respective rates per cent., but for the same number of years. In the present instance the rate of accumulation remains unaltered, but the period of repayment is varied. Consequently the ratio is expressed in terms of the amounts of £1 per annum for the respective unexpired and substituted periods of repayment, but at the same rate per cent. of accumulation.

In the formula in Chapter XXIII relating to a variation in the rate of accumulation, the numerator is the amount of £1 per annum for the unexpired period at the past rate of accumulation, and the denominator is the amount of £1 per annum for the same unexpired period at the future rate of accumulation, thus taking as the basis of the ratio the varying rates of accumulation. But as the formula about to be ascertained depends as to its ratio upon the varying periods of repayment, and there is not any variation in the rate of accumulation, the numerator becomes the amount of £1 per annum at the rate of accumulation common to the two periods for the unexpired period, and the denominator becomes the amount of £1 per annum at the same rate of accumulation for the substituted period. Substituting the above terms for those in the previous formula, the amended formula is ascertained for dealing with problems involving variations in the period of repayment only, but not at the same time involving any variation in the rate of accumulation. The rule and formula as to a variation in the period of repayment only will be expressed in the same abbreviated terms used in Chapter XXIII, dealing with a variation in the rate of accumulation, and these abbreviated terms should be carefully considered. They are fully explained at the head of Chapter XXII. In this case there is not any variation in the rate of accumulation, consequently the past and future rates are the same, and are, in effect, the past rate. This is important when considering a variation in the period of repayment only, or a concurrent variation in the rate of accumulation as well as in the period of repayment.

It is therefore necessary to use the term "future rate" in the after consideration of this and the formula relating to the dual variation in rate and period.

The rule as to a variation in the period of repayment only, the annual increment (ratio) method, is stated in full at the head of this chapter.

As stated in Chapter XXIII, the above rule is sufficiently explicit, but as it will be necessary in the following chapter to combine it with the previous rule relating to a variation in the rate of accumulation it will be expressed as a formula, as follows:—

VARIATION IN THE PERIOD OF REPAYMENT.

The Annual Increment (ratio) Method.

$$\left\{\begin{array}{c}\text{Present}\\ \text{annual}\\ \text{increment}\end{array}\right\} \times \left\{\dfrac{\begin{array}{c}\textit{Amount of £1 per annum}\\ \textit{at future rate}\\ \textit{for unexpired period}\end{array}}{\begin{array}{c}\text{Amount of £1 per annum}\\ \text{at future rate}\\ \text{for substituted period}\end{array}}\right\} = \left\{\begin{array}{c}\text{Future}\\ \text{or}\\ \text{amended}\\ \text{annual}\\ \text{increment}\end{array}\right\}$$

The amounts of £1 per annum in the above rule and formula are at the same rate per cent. of accumulation, but are for varying numbers of years. In this case the future rate is the same as the past rate. Calculation XXIV. C. may now be stated in terms of the above formula, as follows:—

$$1027{\cdot}882 \times \left(\frac{16{\cdot}11303}{9{\cdot}05168}\right) = 1829{\cdot}744$$

and in Chapter XXVI this calculation will be combined with the similar calculation in Chapter XXIII.

The above rule and formula will apply equally to an increase or reduction in the period of repayment, and it is important to remember that an increase in the period will have the effect of reducing the annual instalment to be charged to revenue or rate account in future years. When considering the rate of accumulation in Chapter XXIII it was found that an increase in the rate of accumulation will reduce the annual instalment in future years. In the following chapter (XXVI) the above formula, relating to a variation in the period of repayment will be combined with the formula found in Chapter XXIII, relating to a variation in the rate of accumulation, for the purpose of deriving therefrom a formula which may be applied to a concurrent variation in the period of repayment and the rate of accumulation.

It will be noticed that the numerator in the above formula, relating to the period, is the same as the denominator in the formula in Chapter XXIII, relating to the rate per cent.

CHAPTER XXVI.

SINKING FUND PROBLEMS, RELATING TO THE RATE PER CENT. OF ACCUMULATION AND THE REDEMPTION PERIOD IN COMBINATION.

SUMMARY OF METHODS. GENERAL CONSIDERATIONS. THE METHODS OF ASCERTAINING THE AMENDED ANNUAL INSTALMENT DUE TO A VARIATION IN BOTH THE ABOVE FACTORS IN COMBINATION. THE DEDUCTIVE METHOD, THE ANNUAL INCREMENT (RATIO) METHOD, AND THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. STATEMENT SHOWING THE FINAL REPAYMENT OF THE LOAN BY THE OPERATION OF THE AMENDED ANNUAL INSTALMENT.

DERIVATION OF A RULE AND FORMULA RELATING TO A DUAL VARIATION OF THIS NATURE BASED UPON THE FOREGOING RESULTS, BY THE ANNUAL INCREMENT (RATIO) METHOD.

Summary of the methods of adjustment.

(I) The deductive method, as summarised at the head of Chapter XXIV; which may be compared with the method summarised at the head of Chapter XIX.

Statement XXVI. A.

(II) The direct method, without calculation, as summarised at the head of Chapter XX, will not apply to this variation.

(III) The annual increment (balance of loan) method, as summarised at the head of Chapter XXII.

Statement XXVI. H.

(IV) The annual increment (ratio) method, as summarised below. *Statement XXVI. C.*

Note. The terms used in the following summary are fully discussed at the head of Chapter XXII. It is imperative, in using the above methods, that the future rate of accumulation and the rate of income from the present investments shall be uniform during the whole of the substituted period of repayment.

The Annual Increment (ratio) Method.

The rule as to a concurrent variation in the rate of accumulation, as well as in the period of repayment, may be stated as follows, using the terms explained at the head of Chapter XXII. *Statement XXVI. C.*

Rule. *To find the amended annual instalment to be set aside, and added to the existing sinking fund,*

to be accumulated in future at a rate per cent. greater or less than the rate at which the present annual instalment was calculated

(the future rate),

and to be set aside for a reduced or increased number of years, as compared with the unexpired portion of the original repayment period

(the substituted period).

Proceed as follows:—

(1) *Ascertain the present annual increment of the fund, as described in Chapter XXII.*

(2) *Multiply the annual increment so found by the amount of £1 per annum at the past rate for the unexpired period.*

(3) *Divide the above product by the amount of £1 per annum, at the future rate for the substituted period.*

(4) *The amount so found will represent the future or amended annual increment of the fund under the new conditions. The amended annual sinking fund instalment may be found by deducting therefrom the future annual income from the present investments representing the fund.*

(5) *Prepare a statement showing the final repayment of the loan by the operation of the fund under the amended conditions.* *Statement XXVI. B.*

(6) *Prepare the usual pro forma account previously recommended.* *Pro forma Account, No. 11.*

General Considerations. The predominant factor in all problems of this nature is the variation in the period of repayment because its effect upon the amended annual instalment is far greater than that due to the variation in the rate

of accumulation which will generally lie within very narrow limits. For the reasons given in Chapter XXIV, a variation of this two-fold nature will rarely arise in connection with any individual loan of a local authority, and if such a problem arises in connection with the consolidation of several such loans it will be complicated by other factors which will render necessary a different mode of treatment, as will be explained in Chapter XXXII, dealing generally with the equation of the period of repayment.

The principal application of the methods to be discussed in this chapter will relate to the sinking funds of commercial and financial undertakings, and all the general considerations as to a variation in the period of repayment only, stated in Chapter XXIV, will apply to this example without further reference or amplification.

In dealing with problems which may arise in connection with the sinking funds of local authorities and commercial and financial undertakings, the following important factors have already been discussed, namely:—

1. The amount in the fund.
2. The rate per cent.—
    (*a*) of income upon the present investments.
    (*b*) of future accumulation.
3. The period of repayment.

In discussing the problems relating solely to a variation in the rate per cent. or the period of repayment, in each case there has been combined in one factor, "the annual increment," (1) the original or amended annual instalment, and (2) the past or future income arising from the present investments representing the fund at the time the variation occurs in the rate or period.

This annual increment is fully discussed and described in Chapters XIV and XXII. The majority of the examples used to illustrate the above problems relate to a sinking fund to repay a loan of £26,495 at the end of 25 years, and it has been assumed that the variation, and the consequent necessity for adjustment, occurs at the end of the 12th year in each case. As regards a variation in the rate per cent., it has been proved that the problem may be confined, so far as the actuarial calculation is concerned, to the rate of accumulation. It has also been ascertained that there is a simple ratio existing

between the original and amended annual increments due to a variation in both the rate and the period, and that this ratio is based, not upon the respective rates per cent. of accumulation or upon the number of years in the period of repayment, but upon the respective amounts of £1 per annum as follows:—

1. In the case of a variation in the rate of accumulation, upon the amounts of £1 per annum for the same period of repayment, but at the respective rates per cent. of accumulation. Chapter XXIII.

2. In the case of a variation in the period of repayment, upon the amounts of £1 per annum at the same rate per cent. of accumulation, but for the respective periods of repayment. Chapter XXV.

In the case of variations in the rate per cent. the necessary adjustment has been made, in the first instance, by the deductive method, fully described in Chapter XIX, based upon the whole of the factors governing the fund, after which the result so obtained has been verified by the annual increment (ratio) method based upon the annual increment, as described in Chapter XXII. These results have been utilised to deduce a rule, and a formula expressing the rule, which is fully described in Chapter XXIII.

The enquiry was then extended in a similar manner to an adjustment rendered necessary by a variation in the period of repayment which was considered in Chapters XXIV and XXV, and a similar rule and formula was deduced. In each case it was found that the methods applied equally to an increase or a reduction in the rate of accumulation or period of repayment.

The adjustment consequent upon a dual variation in the rate of accumulation, as well as in the period of repayment, will be fully considered in the present chapter, using the whole of the methods already described, after which a rule and formula relating to the adjustment will be deduced from the results so obtained.

THE DEDUCTIVE METHOD. The present enquiry will also be illustrated by a sinking fund to repay a loan of £26,495 at the end of 25 years, but with a rate of accumulation of 3 per cent., requiring an annual instalment of £712·826 to be set aside for the remaining 13 years. This has been ascertained in Chapter XIX, Statement XIX. A.

The necessity to make the adjustment arises at the end of the 12th year, at which time the amount in the fund is £9932·74, which is represented by investments valued at that

amount, bringing in an annual income at $3\frac{1}{2}$ per cent. per annum of £347·648, and it will be assumed that this income is assured for the whole of the unexpired portion of the original repayment period. At the end of the 12th year, this period is for some reason reduced from 13 years to 8 years, and the rate of accumulation is increased from 3 to $3\frac{1}{2}$ per cent., as in the original conditions in Chapter XV.

The effect will be that the annual instalment of £712·826 will be increased in consequence of the reduction of the period of repayment, but it will not be increased to such an amount as it would have been if the rate of accumulation had remained at 3 per cent., and had not been increased to $3\frac{1}{2}$ per cent. This will be shown later in this chapter in detail, where the amended annual instalment will be divided between these factors. As in previous examples by the deductive method, all the present factors of the fund will be reduced to equivalent amounts of original loan which they will each provide by accumulation at the future rate of $3\frac{1}{2}$ per cent., at the end of the substituted period of 8 years, in order to ascertain, by deduction, the portion of original loan which will remain to be provided by the future accumulation of an additional annual instalment to be charged to revenue or rate or deducted from profits, and a final calculation will be made to ascertain such additional instalment.

This is fully shown in Statement XXVI. A., which is similar in principle to previous statements illustrating the deductive method. This statement shows that the reduction in the original period of repayment from 25 to 20 years (but with an increase in the rate of accumulation) taking place at the end of the 12th year results in an increased annual burden of £769·270 chargeable against the rate account or the revenue account of the undertaking.

It is now possible to review the operation of the sinking fund under the altered conditions in order to ascertain that the amended annual instalment so found will carry out the purpose of the fund, namely, to repay the loan of £26,495, at the end of 20 instead of 25 years. This is shown in Statement XXVI. B., which is exactly similar in principle to the previous statements prepared to illustrate the accuracy of the amended annual instalments found by the deductive and other methods.

THE ANNUAL INCREMENT (RATIO) METHOD. (Rule and Formula.) In Chapter XXII (a variation in the rate of accumulation) as well as in Chapter XXIV (a variation in the period of repayment) the actual adjustment has been made by

the annual increment (ratio) method there described, and from the results so obtained the formula relating to the method has been deduced. In both these variations the ratio is a simple one, depending upon the respective amounts of £1 per annum at the varying rates per cent. in one case and for the varying periods in the other. As the rules and formulæ relating to the above variations have been already ascertained, it is only necessary in the present instance to revert to those formulæ in order to deduce therefrom a modified formula relating to a combination of the above causes of adjustment, and afterwards to make the calculation in the manner shown in Chapters XXII. and XXIV. It would appear from the above theoretical considerations that the two formulæ may be combined in order to deduce therefrom a simple formula which will apply to all problems involving a dual variation in the rate of accumulation and the period of repayment. It is therefore necessary to combine the formula relating to a variation in the rate of accumulation given in Chapter XXIII with that relating to a variation in the period of repayment in Chapter XXV. The factors required are:—

1. The present annual increment.
2. The past and future rates of accumulation.
3. The unexpired and substituted periods of repayment.

For the purpose of the following adjustment the present annual increment, which is the basis of the calculation, is made up as follows:—

Original annual instalment (Statement XIX. A.) ...	£712·826
Income from present investments	347·648
	£1060·474

The above annual income from the present investments, as in all adjustments made by this method, is the amount which has been received in the past, and is not the amount of income which will be yielded thereby during the unexpired or substituted period of repayment. This is one of the fundamental principles of the annual increment (ratio) method, as fully explained in the opening paragraphs of Chapter XXII.

The method of making the adjustment by this method is shown in Calculation XXVI. C., at the end of this chapter.

In each of the examples discussed in Chapters XXII and XXIV the original annual increment was multiplied by the

fractional ratio of £1 per annum. It is therefore obvious that a combination of the above formulæ to relate to the dual variation under discussion must be made by multiplying the present annual increment by each fractional ratio in succession. As already pointed out, the numerator in the fractional ratio relating to the period of repayment is the same as the denominator in the fractional ratio relating to the rate of accumulation, which will cancel out when the respective formulæ are multiplied together; therefore the product of these fractional ratios will consist of the numerator of the ratio relating to the rate of accumulation and the denominator of the ratio relating to the period of repayment as follows:—

Variation in the Rate of Accumulation and the Period of Repayment.

The Annual Increment (ratio) Method.

Variation in Rate. Chapter XXIII. — *Variation in Period Chapter XXV.*

$$\left\{\begin{matrix}\text{Present}\\ \text{annual}\\ \text{increment}\end{matrix}\right\} \times \left\{\frac{\begin{matrix}\text{Amount of £1 per ann.}\\ \text{at past rate,}\\ \text{for unexpired period}\end{matrix}}{\begin{matrix}\textit{Amount of £1 per ann.}\\ \textit{at future rate,}\\ \textit{for unexpired period}\end{matrix}}\right\} \times \left\{\frac{\begin{matrix}\textit{Amount of £1 per ann.}\\ \textit{at future rate,}\\ \textit{for unexpired period}\end{matrix}}{\begin{matrix}\text{Amount of £1 per ann.}\\ \text{at future rate,}\\ \text{for substituted period}\end{matrix}}\right\} = \left\{\begin{matrix}\text{Future}\\ \text{or}\\ \text{amended}\\ \text{annual}\\ \text{increment}\end{matrix}\right\}$$

Note. The factors in the above formulæ which are printed in italics are common to both and will cancel out in the multiplication. Calculation XXVI. C. may now be stated in terms of the above formula in a similar manner to that adopted in Chapters XXIII and XXV:—

$$1060{\cdot}474 \times \left\{\frac{15{\cdot}61779}{16{\cdot}11303}\right\} \times \left\{\frac{16{\cdot}11303}{9{\cdot}05168}\right\} = 1829{\cdot}744$$

The result is the following simplified formula relating to a concurrent variation in the rate of accumulation and the period of repayment:—

Variation in the Rate of Accumulation and the Period of Repayment.

The Annual Increment (ratio) Method.

$$\left\{\begin{array}{c}\text{Present}\\ \text{annual}\\ \text{increment}\end{array}\right\} \times \left\{\frac{\begin{array}{c}\text{Amount of £1 per annum}\\ \text{at past rate,}\\ \text{for unexpired period}\end{array}}{\begin{array}{c}\text{Amount of £1 per annum}\\ \text{at future rate,}\\ \text{for substituted period}\end{array}}\right\} = \left\{\begin{array}{c}\text{Future}\\ \text{or}\\ \text{amended}\\ \text{annual}\\ \text{increment}\end{array}\right\}$$

The amounts of £1 per annum in the above formula are at varying rates per cent., and are also for different numbers of years.

Calculation XXVI. C. will now be expressed in terms of the above formula as follows:—

$$1060{\cdot}474 \times \left\{\frac{15{\cdot}61779}{9{\cdot}05168}\right\} = 1829{\cdot}744$$

It is now possible to state a rule based upon the foregoing formula, using the abbreviated terms set out in full at the head of Chapter XXII, and explained in Chapter XXIII, dealing with the rule relating to a variation in the rate of accumulation. The same terms are used in Chapter XXV, in the rule relating to a variation in the period of repayment. The rule relating to the variation under review is stated in full at the head of this chapter.

Proof of the above Method. The foregoing results which have been obtained by taking both variations into account will now be proved, and the effect of each variation will be shown separately, beginning with the variation in the period of repayment. In Chapter XXIV, an adjustment was made in the annual instalment consequent upon a reduction in the period of repayment from 13 to 8 years, but without any variation in the rate of accumulation. This reduction in the period involved an ultimate deficiency of loan of £7258·21, requiring an additional annual instalment of £801·862 to be set aside for the substituted period of 8 years, as shown in Statement XXIV. A. The accuracy of the calculation was proved by dividing the deficiency in the amount of loan, £7258·21, between the reduced accumulation of the present investments, £2454·85, and of the original annual instalment, £4803·36.

The future deficiency in the accumulation of the present investments, £2454·85, was also reduced to terms of the annual income to arise therefrom.

Although the same method of proof may be applied to the present example the problem will be reduced to terms of the present annual increment of £1060·474, and by deducting therefrom the income from investments, £347·648, included therein, it will be possible at the same time to express in figures the effect upon the additional annual instalment of the reduction in the period of repayment, as distinguished from the effect of the increase in the rate of accumulation.

The calculation will be made by the annual increment (ratio) method, which is the most convenient for the purpose. The problem will be divided into two parts in order to ascertain in the first place the amended annual increment due to the reduction in the period of repayment only, on the assumption that the rate of accumulation remained the same, namely, 3 per cent. This amended annual increment, as shown by the following Statement XXVI. D., is £1862·532, requiring an additional annual instalment of £802·058.

Although it will be necessary to consider the above additional annual instalment of £802·058 later in this chapter, the proof will be continued by taking up the above amended annual increment of £1862·532 in order to ascertain the reduction therein due to the increase in the rate of accumulation from 3 per cent. to $3\frac{1}{2}$ per cent. The calculation cannot be made in terms of the above additional annual instalment of £802·058 for the reasons given in Chapter XXII, Calculation XXII. E., because the benefit of the accumulation of the income from the present investments at the increased rate of accumulation would be lost. The calculation might be made in terms of each of the above factors, namely the annual instalment and the income from investments composing the annual increment of £1862·532, but this would involve only increased labour without any corresponding advantage, seeing that the accuracy of the calculation may be proved by comparing the additional annual instalment to be obtained with that found by the deductive method, and also by comparing the amended annual increment with that found previously by the annual increment (ratio) method, Calculation XXVI. C. This method of proof shows the advantage of the annual increment as a factor even in cases where there is not any variation in the rate of income from the present invest-

a rate of accumulation of $3\frac{1}{2}$ per cent., as in the previous example. (See Statement XIX. A.)

In that case it was found by the deductive method (Statement XXIV. A) that the ultimate deficiency in the amount of loan to be provided, in consequence of the reduction in the period of repayment, was £7258·21, which requires, as shown by Calculation (XXIV) 3, an additional annual instalment of £801·862 to be accumulated at $3\frac{1}{2}$ per cent. for 8 years.

It is therefore necessary to ascertain the equivalent annual instalment to provide the same amount of loan, £7258·21, at the end of 8 years, but to be accumulated at 3 per cent. instead of $3\frac{1}{2}$ per cent. This is shown to be £816·232 in Statement XXVI. F.

Statement XXVI. G. shows that if the additional annual instalment of £816·232, as found by Calculation XXVI. F. be adopted, there will, at the end of the substituted period of 8 years, be in the fund an amount of £126·04 in excess of the amount actually required to repay the loan; and therefore that the additional annual instalment of £816·232 must be reduced by an annual sum which, if accumulated at 3 per cent., will amount to £126·04 at the end of 8 years, which is the annual sinking fund instalment which will provide, or the annuity which will amount to that sum, under the above conditions. By Calculation (XXVI) 5 the annual sum is found to be £14·174.

The correct additional annual instalment required for the purpose of showing the separate effect of the variation in the period therefore is:—

The above calculated instalment of	£816·232
reduced by the above annuity of	£14·174
leaving the actual additional annual instalment of	£802·058

which agrees with the amount found by Statement XXVI. D. by the annual increment (ratio) method.

THE ANNUAL INCREMENT (BALANCE OF LOAN) METHOD. In previous chapters attention has been directed to the principles underlying this method. It resembles very closely the practice adopted by such local authorities as are able to apply the whole of the annual instalments towards the immediate actual redemption of debt. In such cases the interest upon the debt

so redeemed, and the future annual instalments, constitute the annual increment of this method provided the rate of interest upon such redeemed debt is the same as the rate of accumulation. In case there is any variation in these two rates per cent. the annual difference may be transferred as and when it arises to the debit or credit of the revenue or rate account. It is an essential principle of this method that the resulting annual instalment, the future rate of income from the present investments, and the rate of accumulation shall continue without variation during the whole of the unexpired portion of the repayment period. Any departure from uniformity in these respects has already been pointed out. Chapter XVI, dealing with the adjustment of a deficiency in the fund contains full particulars of the method of finding the additional annual instalment to be spread over a portion only of the unexpired repayment period, and Chapter XXVII explains the method of correcting the annual instalment in consequence of a variation in the future rate of income to be received from the present investments, which is expected to occur at a future date during such unexpired period, and a similar method of adjustment will apply to a variation in the rate of accumulation occurring at such a future date.

In dealing with such a future variation in the rate per cent. of income or of accumulation in Chapter XXVII, a distinction has been drawn between a reduction which, although anticipated, is uncertain both as to rate and time, and one in which both factors are definite, as was the case with the reduction in the dividend on consols under Mr. Goschen's Finance Act, 1888. In Chapter XXVII attention is also directed to the difference between the arithmetical and true mathematical methods of arriving at the equated rate per cent., and it is there pointed out that the same difference in such methods occurs in the equation of the period of repayment, as will be fully described in Chapter XXXII. Any variations in the above factors of rate per cent. or period of repayment anticipated to arise during the unexpired portion of the repayment period, whether definite or estimated, are met by finding the amount of an annuity by the method "by step," fully described in Chapters XVI and XXVII, both of which contain a description of the longer as well as the simplified method of such calculation.

The following Statement, XXVI. H., shows the method of proving the previous results by the annual increment (balance of loan) method.

The Redemption Period and The Rate per cent. Statement XXVI. A.

The Deductive Method.

Showing the method of adjusting the annual instalment in consequence of a variation in the period of repayment accompanied by a variation in the rate of accumulation, the rate of income from the present investments being unaltered and being the same as the future rate of accumulation. If these rates are unequal or are varied proceed as in Chapter XIX.

Conditions before adjustment (at end of 12th year),

Amount of original loan repayable in 25 years ...	£26,495
Amount in the fund (at end of 12th year)	£9932·74
Present annual income (previously) received therefrom at 3½ per cent. per annum	£347·648
Present annual instalment, to be accumulated for 13 years, at 3 per cent.	£712·826
Present annual increment	£1060·474

Variation from the above conditions :—

The period during which the loan shall be redeemed is reduced from 13 to 8 years, and

the rate of accumulation of the fund is increased from 3 to 3½ per cent.

The substituted period of repayment ... 8 years.

The future rate of accumulation ... ... 3½ per cent.

		Equivalent amount of original loan.
Present investments (at end of 12th year)	£9932·74	
Amount thereof, accumulated for 8 years at 3½ per cent. Calculation (XXIV) 1		£13079·53
Present annual instalment	£712·826	
Amount thereof, accumulated for 8 years at 3½ per cent. Calculation (XXVI) 1		£6452·28
Provision already made will, at the end of 8 years, repay loan of		£19531·81
Additional annual instalment required :—		
Balance, being amount of original loan unprovided for owing to the above reduction in the period of repayment from 13 to 8 years, but reduced in consequence of the increase in the rate of accumulation from 3 to 3½ per cent., requiring an additional annual instalment to be set aside and accumulated for 8 years at 3½ per cent. ...		£6963·19
Additional annual instalment Calculation (XXVI) 2	£769·270	
Amount of original loan		£26495·00

Amended annual increment, being :—

Annual income from investments ...	£347·648
Amended annual instalment	1482·096
	£1829·744

The Redemption Period and The Rate per cent. Statement XXVI. B.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the sinking fund, after making the adjustment in the annual instalment consequent upon a variation in the period of repayment accompanied by a variation in the rate of accumulation.

		Equivalent amount of original loan.
Present investments (at end of 12th year)		£9932·74
Amended annual increment:—		
Present annual instalment	£712·826	
Additional annual instalment ...	769·270	
Total out of revenue	£1482·096	
Income from investments	347·648	
	£1829·744	
Amount thereof, accumulated for 8 years at $3\frac{1}{2}$ per cent. Calculation (XXIV) 6		£16562·26
Amount of original loan		£26495·00
Amended annual instalment ...	£1482·096	

The Redemption Period and The Rate per cent. — Calculation XXVI. C.

The Annual Increment (ratio) Method.

To find the amended annual increment (and therefrom the amended annual instalment) in a sinking fund, in which there is a variation in the period of repayment accompanied by a variation in the rate of accumulation, with or without any variation in the rate of income upon the present investments.

Required the annual increment to be accumulated for a period of 8 years at $3\frac{1}{2}$ per cent., which is equivalent to an annual increment of £1060·474, to be accumulated for a period of 13 years at 3 per cent.

$$1060{\cdot}474\left\{\frac{\text{Amount of £1 per ann., 13 years, 3\%}}{\text{Amount of £1 per ann., 8 years, }3\tfrac{1}{2}\%}\right\}=1829{\cdot}744$$

or by Table III, giving the amounts of £1 per annum

$$\frac{1060{\cdot}474\times 15{\cdot}61779}{9{\cdot}05168}=1829{\cdot}744$$

Log. Present annual increment ...	1060·474	3·0255000
add Log. Amount of £1 per annum Table III, 13 years, 3 per cent.	15·61779	1·1936196
	16562·26	4·2191196
deduct Log. Amount of £1 per annum Table III, 8 years, $3\frac{1}{2}$ per cent.	9·05169	0·9567296
Log. Amended annual increment		3·2623900

Amended annual increment		1829·744
To find the amended annual instalment:—		
deduct the income from investments, $3\frac{1}{2}$ per cent.		347·648
Amended annual instalment		1482·096
being Present annual instalment ...	712·826	
Additional annual instalment	769·270	
		1482·096

The Redemption Period and The Rate per cent. — Statement XXVI. D.

The Annual Increment (ratio) Method.

Required the annual increment to be accumulated for a period of 8 years at 3 per cent., which is equivalent to an annual increment of £1060·474, to be accumulated for a period of 13 years, also at 3 per cent.

To show the separate effect of the variation in the period.

Present annual increment	1060·474	3·0255000
multiply by amount of £1 per annum, 13 years, 3 per cent.	15·61779	1·1936196
	16562·26	4·2191196
divide by amount of £1 per annum, 8 years, 3 per cent. ...	8·89234	0·9490159
Log. Amended annual increment		3·2701037
Amended annual increment		1862·532
being Income from investments ...	347·648	
Present annual instalment ...	712·826	
Additional annual instalment	802·058	
		1862·532

The Redemption Period and The Rate per cent. Statement XXVI. E.

The Annual Increment (ratio) Method.

Required the annual increment to be accumulated for a period of 8 years at $3\frac{1}{2}$ per cent., which is equivalent to an annual increment of £1862·532, to be accumulated for a like period of 8 years, but at 3 per cent.

To show the separate effect of the variation in the rate per cent.

Present annual increment	1862·532	3·2701037
multiply by amount of £1 per annum, 8 years, 3 per cent. ...	8·89234	0·9490159
	16562·26	4·2191196
divide by amount of £1 per annum, 8 years, $3\frac{1}{2}$ per cent. ...	9·05169	0·9567296
Log. amended annual increment		3·2623900
Amended annual increment		1829·744
being Income from investments ...	347·648	
Present annual instalment ...	712·826	
Additional annual instalment	769·270	
		1829·744

The Redemption Period and The Rate per cent. — Statement XXVI. F.

The Annual Increment (ratio) Method.

Required the annual instalment to be accumulated for a period of 8 years at 3 per cent., which is equivalent to an annual instalment (as in XXIV. A.) of £801·862, to be accumulated for a like period of 8 years, but at 3½ per cent.

Present annual instalment	801·862	2·9040988
multiply by amount of £1 per annum, 8 years, 3½ per cent. ...	9·05169	0·9567296
	7258·21	3·8608284
divide by amount of £1 per annum, 8 years, 3 per cent. ...	8·89234	0·9490159
Log. amended annual instalment		2·9118125
Amended annual instalment ...		816·232

The Repayment Period and The Rate per cent. — Statement XXVI. G.

The Deductive Method.

Showing for purpose of proof only the surplus which will arise in the fund by adopting the additional annual instalment of £816·232 found in Statement XXVI. F., instead of the instalment of £802·058 in Statement XXVI. D. The correction of this surplus is shown below.

		Equivalent amount of original loan.
Present investments		£9932·74
Present annual increment	£1060·474	
Amount thereof, accumulated for 8 years at 3 per cent. Calculation (XXVI) 3		£9430·09
Amended annual instalment, as in XXVI. F.	£816·232	
Amount thereof, accumulated for 8 years at 3 per cent. Calculation (XXVI) 4		£7258·21
Amount in the fund at end of 8 years		£26621·04
Amount of original loan		26495·00
Surplus		£126·04
Annual instalment to provide £126·04 at the end of 8 years at 3 per cent. Calculation (XXVI) 5		£14·174
being the annual instalment, as shown in Statement XXVI. F....	£816·232	
less the annual instalment, as shown in Statement XXVI. D.	£802·058	
		£14·174

The Repayment Period and The Rate per cent. **Statement XXVI. H.**

The Annual Increment (balance of loan) Method.

To find the amended annual sinking fund instalment consequent upon a variation in the period of repayment, accompanied by a variation in the rate of accumulation.

For Rule, see Chapter XXII.

Amount of original loan		£26495·00
deduct amount in the fund at the end of the 12th year		£9932·74
Balance of loan		£16562·26
Amended annual increment, to be added to the fund, and accumulated at $3\frac{1}{2}$ per cent., to provide this amount at the end of 8 years Calculation (XXIV) 6		£1829·744
deduct income to be received from the present investments, £9932·74, at $3\frac{1}{2}$ per cent. ...		£347·648
Amended annual instalment		£1482·096
being Present annual instalment ...	£712·826	
Additional annual instalment	769·270	
		£1482·096

Pro forma Sinking Fund Account No. 11.

A Variation in the Redemption Period, and in the Rate of Accumulation.

Loan of £26,495, repayable at the end of 25 years.

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the increased annual instalment of £1482·096.

Statement XXVI. B. Rate of accumulation, $3\frac{1}{2}$ per cent.

The amount in the fund at the end of the 12th year, £9932·744, is the correct calculated amount, as shown by Calculation (XV) 2, and by the pro forma account, No. 1, Chapter XV.

Year.	Amount in the fund at beginning of year.	Income received from investments $3\frac{1}{2}$ per cent.	Annual sinking fund instalments	Amount in the fund at end of year.	Year.
1					1
2					2
3					3
4					4
5					5
6					6
7					7
8					8
9					9
10					10
11					11
12				9932·744	12
13	9932·744	347·648	1482·096	11762·488	13
14	11762·488	411·687	1482·096	13656·271	14
15	13656·271	477·969	1482·096	15616·336	15
16	15616·336	546·572	1482·096	17645·004	16
17	17645·004	617·575	1482·096	19744·675	17
18	19744·675	691·063	1482·096	21917·834	18
19	21917·834	767·124	1482·096	24167·054	19
20	24167·054	845·850	1482·096	26495·000	20
21					21
22					22
23					23
24					24
25					25

CHAPTER XXVII.

SINKING FUND PROBLEMS, RELATING TO THE RATE PER CENT. OF INCOME UPON THE PRESENT INVESTMENTS REPRESENTING THE FUND (*in continuation of Chapter XX*).

In which the rate of income yielded by such investments is not uniform during the whole of the unexpired portion of the repayment period.

A. In which the future variation in the rate of income is known, and is definite, both as to time and amount. Statement XXVII. A.

B. In which the future variation in the rate of income is anticipated, but is uncertain, both as to time and amount. Statement XXVII. D.

Summary of methods. How the variation may arise and the general considerations applicable thereto. The deductive method. The annual increment (balance of loan) method. Statement showing the final repayment of the loan by the operation of the amended annual instalment. Comparison with Variation A (rate of income), in Chapter XX, where the rate is uniform during the whole period. Calculation of the equated annual income by the arithmetical method and demonstration of the error involved.

Summary of the methods of adjustment.

(I) The deductive method, (A) as summarised below. *Statement XXVII. A.*

(II) The direct method, without calculation, as summarised at the head of Chapter XX, will not apply.

(III) The annual increment (balance of loan) method, as summarised at the head of Chapter XXII, will apply after finding the equated annual income by the deductive method (B), summarised below. *Statement XXVII. D.*

(IV) The annual increment (ratio) method, as summarised at the heads of Chapters XXIII, XXV, and XXVI, will not apply, as there is not any variation in the rate of accumulation.

SUMMARY OF THE DEDUCTIVE METHOD, (A) *of ascertaining the future or amended annual sinking fund instalment due to a variation in the rate of income to be received upon the present investments representing the fund when it is known at the time of making the adjustment that such future rate of income will not be uniform during the unexpired period of repayment, but will be varied by a definite amount at a known future date. In this case the problem is not complicated by any variation in the rate of accumulation or the period of repayment.*

Statement XXVII. A.

The terms used in the following summary are fully explained at the head of Chapter XXII. The unexpired period of repayment is divided into two known parts, namely:—

The first period, during which the rate of income upon the present investments will remain unaltered.

The second period, during which the rate of income upon the present investments will be varied by a known amount.

Memo. (A). If the rate of income be varied at the time of making the adjustment, as well as at a known future date, adjust the annual instalment, as described in Chapter XX, before operation (6) following.

(1) Having ascertained the value of the present investments in the manner already described, calculate the annual amount of income (at each rate per cent.) to be received during the first and second periods respectively.

(2) Calculate the amount of an annuity, equal to the annual income to be received during the first period at the original rate of income, for the number of years in that period, at the future rate of accumulation.

Calculation (XXVII) 1.

(3) Calculate the sum to which the amount so found in (2) will accumulate at the end of the number of years in the second period at the future rate of accumulation.

Calculation (XXVII) 2.

(4) *Calculate the amount of an annuity, equal to the annual income to be received during the second period at the reduced rate of income, for the number of years in that period at the future rate of accumulation.*
Calculation (XXVII) 3.

(5) *The amount found in (3) added to the amount found in (4) will give the accumulated amount of the income from investments at the end of the unexpired period of repayment, expressed in terms of original loan.*
[Here refer to Memo. A, above.]

(6) *Calculate the accumulated amount of the original or amended annual instalment for the total number of years in the unexpired repayment period at the future rate of accumulation, expressed in terms of original loan.* *Calculation (XIX) 2.*

(7) *Add together the amounts found in (5) and (6) and the value of the present investments found in (1), and deduct the total from the amount of the original loan.*

(8) *The difference will be the amount of loan unprovided for in consequence of the above decrease in the rate of income upon the present investments during the second period.*

(9) *Calculate the annual instalment which will provide the amount of loan found in (8) at the end of the total unexpired portion of the repayment period at the future rate of accumulation.* *Calculation (XXVII) 5.*

(10) *The annual instalment found in (9) added to the original or amended annual instalment found in (6) will be the future or amended annual instalment required.*

(11) *Prepare a statement showing the final repayment of the loan by the operation of the fund under the amended conditions.* *Statement XXVII. B.*

(12) *Prepare the usual pro forma account previously recommended.* *Pro form Account, No. 12.*

If the above problem be complicated by a variation in the rate of accumulation, or the period of repayment, or both, it may be solved by a combination of the methods previously demonstrated, but which need not be specially described.

SUMMARY OF THE DEDUCTIVE METHOD (B) *of ascertaining the future equated annual income upon the present investments representing the fund when it is anticipated that the future rate of income will not be uniform during the unexpired period of repayment, but the amount of the variation, and the date at which it will occur, are not known at the time of making the adjustment.*

In this case the problem is not complicated by any variation in the rate of accumulation, or the period of repayment.

The terms used in the following summary are fully explained at the head of Chapter XXII. The unexpired period of repayment is divided into two estimated parts, as follows:—

The first period, during which the present rate of income will continue to be received.

The second period, during which the rate of income is expected to be varied, but the exact amount of such variation can only be estimated.

(1) *Estimate, as accurately as possible, the period during which the present investments will continue to yield the rate of income now received. (The first period.)*

(2) *Deduct the number of years, so estimated, from the unexpired portion of the original repayment period. (The second period.)*

(3) *Estimate as accurately as possible, the rate of income which will be yielded by the present investments during the second period, as ascertained in (2).*

(4) *Ascertain the value of the present investments in the manner already described.*

(5) *Calculate the annual amount of income to be received during the first period, estimated as in (1), at the present unaltered rate of income.*

(6) *Calculate the annual amount of income expected to be received during the second period, as ascertained in (2), at the rate per cent., estimated as in (3).*

(7) *Calculate the accumulated amount of an annuity equal to the annual income to be received during the first period as ascertained in (5) for the number of years in the first period as estimated in (1) at the present unaltered rate of accumulation. Calculation (XXVII) 1.*

(8) *Calculate the sum to which the amount found in (7) will accumulate at the end of the second period found in (2) at the present unaltered rate of accumulation.*
Calculation (XXVII) 2.

(9) *Calculate the accumulated amount of an annuity equal to the annual income estimated to be received, as found in (6) during the second period found in (2) at the rate per cent. of income estimated in (3) at the present unaltered rate of accumulation.* *Calculation (XXVII) 3.*

(10) *The amount found in (8) added to the amount found in (9) will represent the amount of original loan which will be provided at the end of the unexpired period of repayment by the future accumulation, at the unaltered rate, of the annual amounts of income from the present investments found as above,*

as to the first period, in (5).
as to the second period, in (6).

(11) *Calculate in the manner already described, using the author's standard calculation form, No. 3x, Chapter X, the equal annual instalment or annuity which will amount to the total sum found in (10) at the end of the unexpired repayment period, at the unaltered rate of accumulation.* *Calculation (XXVII) 6.*

(12) *The annuity, or equal annual sum, found in (11) is the equated annual income required, and may be treated as part of the future or amended annual increment in all problems involving a variation in the rate per cent. of income upon the present investments accompanied by a variation in the rate of accumulation.*
Pro forma Account, No. 13.

General Considerations. Reference has already been made in previous chapters to the difficulty which arises, especially in cases where the repayment of the loan is spread over long periods, of fixing the future rate of accumulation of the sinking fund, and a similar difficulty will also occur in connection with the future rate of income to be received upon the present investments representing the amount in the fund. In adjustments similar to those under review the future rate of accumulation will nearly always be a matter of speculation and any uncertainty in the matter is met in practice by assuming a rate of accumulation on the low side. The rate of income

to be received in future upon the investments representing the fund at the time of making the adjustment may in some cases be assured for the whole of the unexpired portion of the repayment period, and in Chapters XIX, XX, and XXI, dealing with Variations A, B, and C, it has been assumed that this will be the case in order to simplify the calculation and to demonstrate the principle. It has in fact been assumed that the reduction in the rate of income on the present investments in Variations B and C has been partly the cause of the rectification of the annual instalment. If at any future time the rate of income from the present investments should again be reduced it would be necessary to repeat the adjustment. This reduction in the rate of income yielded by the present investments may be due to a decrease in the rate of interest upon a security similar to a mortgage without any fall in the capital value of the investment, or might be due to the realisation of part of the security at a loss, in which case the additional annual instalment would include the replacement of the deficiency of capital, and a further amount due to the reduced income upon such capital realised, although the actual rate per cent. yielded on the re-investment might remain the same. But the rate of income to be received from the present investments may be reduced at the time of making the adjustment, and at the same time it may also be provided that a further additional reduction shall take place at a fixed future date. These are definite data which may be made the subject of actual calculation. Such an instance occurred in 1888, when, by Mr. Goschen's Finance Act, the rate of interest on Consols was reduced from 3 per cent. to $2\frac{3}{4}$ per cent. for a period of 15 years until 1903, and the Act provided that the interest should be then further reduced to $2\frac{1}{2}$ per cent., the present rate. If the typical Sinking Fund which has been used to illustrate the previous examples had been, in 1888, invested in Consols and had then an unexpired period of 13 years to run, the method of calculation adopted in all the variations already considered would have been accurate, and it would have been quite correct to base the additional annual instalment on an assured yield of $2\frac{3}{4}$ per cent. But if the fund had, in 1888, been invested in Consols, and had then an unexpired period of 20 years to run, the problem would have been very different seeing that the present investments would yield $2\frac{3}{4}$ per cent. for 15 years and $2\frac{1}{2}$ per cent. for the remaining 5 years.

A similar calculation "by step" has already been made when dealing with the adjustment of a deficiency in the fund

by means of an additional annual instalment to be spread over the earlier years only of the unexpired repayment period (see Variation II (Deficiency), Chapter XVI), and a similar procedure will apply to the above conditions. At the end of this chapter the method of ascertaining the amount of an annuity in this way will be further explained and illustrated by a shorter mode of calculation. (Statement XXVII. C.

Illustration of the Method. The method of making the adjustment will be illustrated by means of the results obtained in Chapters XIX and XX. In Variation A, Chapter XIX, the future rate of income upon the present investments is assumed to be $3\frac{1}{2}$ per cent. for the whole of the unexpired period of 13 years, and in Variation B, Chapter XX, to be reduced to 3 per cent. for the same unexpired term. This reduction in the rate of income in Variation B is assumed to take place at the end of the 12th year and to continue unaltered during the remaining 13 years, but a similar change in the rate of income to that in the case of Consols already referred to might take place during the unexpired term of 13 years in Variation A. The present annual increment in Variation A includes income at $3\frac{1}{2}$ per cent. on investments valued at £9932·74, viz., £347·648 per annum which, at the end of the unexpired period of 13 years, will amount at 3 per cent., as shown by Calculation (XIX) 1, to £5429·494.

In Variation B the present annual increment includes income at 3 per cent. on the same investments, viz., £297·984 per annum, and this at the end of the period of 13 years will amount at 3 per cent., as shown by Calculation (XX) 1, to £4653·85.

Both the above annual amounts of income are assumed to accumulate at 3 per cent. for the 13 years so that the question of the rate of accumulation does not affect the problem. But if in Variation A there had been a reduction in the rate of income taking place at the end of the 8th year of the unexpired period of 13 years, the accumulated amount of the annual income at the end of the 13 years would have been different.

Instead of £347·648 per annum at $3\frac{1}{2}$ per cent. for 13 years there would have been:—

Income at $3\frac{1}{2}$ per cent., or £347·648 per annum for 8 years, followed by

Income at 3 per cent., or £297·984 per annum for 5 years,

both accumulating at 3 per cent. for the above periods; and in

addition the accumulation at 3 per cent. of a sum (to which £347·648 per annum will amount at 3 per cent. at the end of 8 years) continued without further annual addition for a period of 5 years.

The amount of loan which will be provided at the end of the period of 13 years by the accumulation of the above income from investments may be ascertained by the following method by "step."

Amount of £347·648 per annum for 8 years accumulated at 3 per cent. Calculation (XXVII) 1	£3091·403
Amount of the above sum of £3091·403 in 5 years accumulated at 3 per cent Calculation (XXVII) 2	£3583·783
Amount of £297·984 per annum for 5 years accumulated at 3 per cent Calculation (XXVII) 3	£1582·037
Accumulated amount at the end of 13 years ...	£5165·820

as compared with the following amounts already ascertained on the assumption that the rate of income will be uniform during the whole period of 13 years:—

at $3\frac{1}{2}$ per cent.	XIX. A.	£5429·494
at 3 per cent.	XX. A.	£4653·850

The above sum of £5165·82 represents the portion of original loan which will be provided at the end of the period of 13 years by the accumulation at 3 per cent. of the income from investments (at $3\frac{1}{2}$ per cent. for the first 8 years and at 3 per cent. for the remaining 5 years).

The above figures show the deficiency in the amount of original loan to be provided by the accumulation of the annual income from investments if such investments had yielded the above definite although variable rates during the period of 13 years, as compared with a uniform rate of $3\frac{1}{2}$ per cent. as assumed in the calculation of the amended annual instalment in Variation A, Chapter XIX. The following Statement, XXVII. A, shows the deductive method of ascertaining the amended annual instalment in consequence of a reduction in the rate of income of the above character.

The original annual instalment as shown in Statement XIX. A. is	£712·826
and the additional annual instalment due to the variation in the rate of income from the present investments now under review, as found by Calculation (XXVII) 5, is	£16·883
or an amended annual instalment as shown by Statement XXVII. A., of	£729·709

The above reduction in the rate of income from the present investments at the end of the 8th year involves a further deficiency of £263·67 in the amount of loan which will be provided at the end of the unexpired repayment period of 13 years, and is the difference between the future accumulation of the income from investments shown in

Statement XIX. A.	£5429·49
and the amount ascertained as above	£5165·82
or	£263·67

requiring a further additional annual instalment of £16·883 as shown by Calculation (XXVII) 5.

THE FUTURE EQUATED ANNUAL INCOME. The future or amended annual increment will now be considered. Statement XIX. A., Variation A, shows that the future or amended annual increment to be accumulated for 13 years at 3 per cent. is £1060·474. This is made up of:—

Amended annual instalment	£712·826
and future annual income from investments at $3\frac{1}{2}$ per cent.	£347·648
	£1060·474

In the case now under consideration the future annual increment will still be £1060·474 seeing that there is not any variation in the rate of accumulation, as proved by the results obtained in Chapter XX, Variation B, but it will be an *equated* and not an actual annual increment, ascertained as follows:—

Present annual instalment, as above...	£712·826
Additional annual instalment due to the reduction in the rate of income from the present investments from $3\frac{1}{2}$ to 3 per cent. during the last five years of the period of repayment Calculation (XXVII) 5	£16·883
Amended annual instalment (Statement XXVII. A.)	£729·709
leaving to be provided, an equated annual amount of income from the present investments	£330·765
Amended annual increment, as above	£1060·474

Under the altered conditions, the actual annual income from the present investments will be £347·648 per annum for 8 years, followed by £297·984 per annum for 5 years, and these annual sums accumulated at 3 per cent. will, at the end of the 13 years, amount together to £5165·82. If the calculation be correct the above equated annual income (£330·765) should represent an equated sinking fund instalment which will provide £5165·82 at the end of 13 years if accumulated at 3 per cent., and this is found to be the case by Calculation (XXVII) 6. The above amount of £330·765 may therefore be described as the true equated annual income, being the annual sum which, accumulated for 13 years at 3 per cent., is equivalent to the two annual amounts of income of £347·648 and £297·984 to be accumulated at 3 per cent. for the successive periods of 8 and 5 years as above described. This equated annual amount of income of £330·765 does not take any part in the actual working of the fund. It is merely the average annual equivalent (over the whole period) of the known actual varying amounts which will be received during the period and is used here merely to demonstrate that the actual successive annual amounts of income are the equivalents of the calculated equated annual income.

Where the future variation in the rate of income during the unexpired portion of the repayment period is definite both as to time and rate per cent. and is not an estimate the above deductive method (A) should be adopted exclusively. This method is summarised at the head of this chapter.

THE METHOD BY STEP. In Chapter XVI, dealing with the correction of a deficiency in a sinking fund, two methods of

adjustment have been shown depending upon the period during which the additional annual instalment is required to be set aside and added to the fund. In Variation I, such additional annual instalment is spread equally over the whole of the unexpired portion of the repayment period of 13 years, and as shown in that chapter the method of adjustment is a simple one.

In Variation II the conditions are more complicated because it is required that the additional annual instalment shall be set aside and added to the fund during the earlier years only of the unexpired portion of the repayment period. This involves an increased annual charge as compared with Variation I, as follows:—

In Variation I the additional annual instalment to be spread equally over the whole of the unexpired repayment period of 13 years, as shown by Statement XVI. A., is	£45·594
In Variation II, the additional annual instalment to be set aside during the first 5 years only of the unexpired repayment period of 13 years, has been obtained by the method "by step" there described, and as shown in Statement XVI. C, is	£104·039

proving that the increased annual burden is due solely to the reduction in the period allowed for the adjustment of the deficiency, the rate of accumulation being the same in both cases. In this example there is a variation from the general rule applicable to the accumulation of a given sum of money now in hand and also of an annual or other periodic sum, or annuity, which rule is based upon a steady and uninterrupted accumulation, and does not provide for any break in any of the factors of rate per cent. or period.

The present example relates to the correction of a sinking fund in consequence of a variation in the future rate of income to be received on the present investments representing the fund, which occurs in the middle of the unexpired portion of the repayment period, and therefore a similar method by "step" may be adopted.

Having ascertained the additional annual instalment of £729·709 in the above manner, Statement XXVII. B. has been prepared showing the final repayment of the loan. This statement shows the accumulated amount, £11106·10 of the annual increment of £1077·357 at the end of the 13 years by the longer method "by step" by two Calculations, (XXVII) 1 and 2,

relating to the annual income of £347·648. The same calculation may be made by the shorter method shown in Statement XXVII. C., described later, which is similar to Calculation XVI. D. 1. The method shown in Statement XXVII. A. is the one which should be adopted in cases where the further reduction in the rate of income from the present investments is not an estimate, but is known and is definite as to the rate per cent. as well as the period. The difference between the arithmetical and true methods of arriving at the equated period of repayment will be fully discussed in Chapter XXXII where it will be shown that the same principle applies, and at the end of this chapter the question of equation as applied to the rate per cent. will be briefly discussed.

Calculations (XXVII) 1 and 2 are made with the object of finding the amount which will be provided at the end of 13 years by the accumulation at 3 per cent. of an annuity of £347·648 to be set aside for the first 8 years of that period when the annuity ceases, but the sum to which it then amounts will continue to accumulate at the same rate for the remaining 5 years.

Calculation (XXVII) 1 shows that at the end of 8 years the annuity of £347·648 will amount to £3091·403, and Calculation (XXVII) 2 shows that this sum accumulated for a further 5 years will amount to £3583·783. The amount of £3091·403 found by Calculation (XXVII) 1 becomes the basis of Calculation (XXVII) 2, but being only an intermediate factor is not of any further interest in the problem. The two calculations may therefore be combined with advantage as shown in Statement XXVII. C., using Thoman's method, as being the simpler, pointing out, however, that if the calculation be required at any rate per cent. not worked out by Thoman, the values of the factors may be ascertained by means of the formulæ already referred to. All that is necessary is to remember that (a^n) of Thoman may be found by means of the following factors referred to in Chapters IX and X.

$$\text{Log } a^n = \text{Log } R^N + \text{Log } r - \text{Log } (R^N - 1)$$

It is important to remember, however, that the method applies equally to cases in which the rate of accumulation is not the same in both periods, and that the periods in each factor R^N may be different. In the following Calculation XXVII. C. 10 has been added to the sum of the logs of the annuity and of R^N, for the reason fully explained in Chapter IX, dealing with Thoman's Tables.

THE ARITHMETICAL METHOD OF FINDING THE EQUATED ANNUAL INCOME. Although the actual amount of the immediate reduction in the future rate of income to be received upon the present investments may be known at the time of making the adjustment, it may be anticipated that there will be a further reduction of an unknown amount at some future date and it may be deemed advisable to make allowance therefor. The above method may be used although in this case there is one known and one estimated rate per cent. of income.

In such a case the amount of the further reduction is problematical, and it is therefore preferable and permissible to use the shorter and more direct arithmetical method of finding the equated annual income to be received over the period of 13 years, and this will now be shown as applied to the above particulars in order to compare the result with the mathematical method just described.

There is (1) an annual income for 8 years at $3\frac{1}{2}$ per cent. of	£347·648
and (2) an annual income for a further 5 years at 3 per cent. of	£297·984

and it is required to find the equated annual income for 13 years, which is equivalent to the above. Proceed as follows:—

(1) 347·648 × 8	=	£2781·184
(2) 297·984 × 5	=	1489·920
		£4271·104
this total, divided by 13, gives an annual sum of		£328·546

which is the arithmetical equivalent of the above annual amounts of £347·648 and £297·984 for the above respective periods.

This calculated annual income of £328·546 upon £9932·74, the value of the present investments, is equivalent to 3·31 per cent., but the actual rate per cent. is immaterial. What is important is the fact that the estimated annual income to be received, calculated in this manner, is only £328·546, as compared with the true equated annual amount of £330·765, ascertained by the mathematical method as above, or a decrease of £2·219 per annum.

Seeing that the total future or amended annual increment must be £1060·474 in order to provide the balance of loan at the above rate of accumulation the estimated annual deficiency of £2·219 must be added to the additional instalment of £16·883 (found as above in Calculation (XXVII) 5), to be charged against revenue or rate, with the result that at the end of the period the fund will be in excess of the proper amount by the accumulation of the larger actual amounts of the future annual income owing to the fact that the actual income received will be in excess of the equated amount assumed. Further, the actual amounts in the fund at the end of each year will exceed the amounts shown by the pro forma account already referred to, by an increasing annual surplus, for the same reason. The difference in this case is only small, but it is proof that the arithmetical method of equation is incorrect, and the extent of the error depends upon the actual rates per cent. of income, the periods during which they operate, and the amount in the fund. The same error will be found in the arithmetical method generally adopted when considering the equation of the period of repayment in Chapter XXXII, and the two results should be carefully compared.

The Rate per cent. Statement XXVII. A.

The Deductive Method.

Showing the method of adjusting the annual instalment, in consequence of a known variation in the rate of income upon the present investments, to occur at a known future date, without any variation in the rate of accumulation or in the period of repayment.

The original conditions in this example are similar to Variation B, in Chapter XX, in which case the reduction in the rate of income, from $3\frac{1}{2}$ to 3 per cent., took effect at the end of the 12th year. In the present instance, however, the reduction in the rate of income, from $3\frac{1}{2}$ to 3 per cent., does not operate immediately, but occurs at the end of 8 years. The future annual income from the present investments will therefore be:—

for 8 years, at $3\frac{1}{2}$ per cent., on £9932·74 ...	£347·648
for 5 years at 3 per cent., on £9932·74 ...	297·984

		Equivalent amount of original loan.
Present investments (at end of 12th year)		£9932·74
Income from present investments :—		
Amount of £347·648 per annum, for 8 years at 3 per cent. Calculation (XXVII) 1	£3091·403	
Amount of £3091·403, in 5 years at 3 per cent. Calculation (XXVII) 2	£3583·783	
Amount of £297·984 per annum, for 5 years at 3 per cent. Calculation (XXVII) 3	£1582·037	
		£5165·82
Present annual instalment (Variation A) :—		
Amount of £712·826 per annum, for 13 years at 3 per cent. Calculation (XIX) 2 (£680·234)	£10623·75	
Calculation (XIX) 3 (£32·592)	509·02	
		£11132·77
Present annual instalment	£712·826	
Provision already made will repay loan of		£26231·33
Additional annual instalment required :—		
Balance, being amount of original loan unprovided for, owing to the above reduction in the rate of income from the present investments during the later years of the unexpired portion of the repayment period, requiring an additional annual instalment to be set aside and accumulated for 13 years at 3 per cent.		£263·67
Additional annual instalment Calculation (XXVII) 5	£16·883	
Amount of original loan		£26495·00
Amended annual instalment	£729·709	

The Rate per cent. **Statement XXVII. B.**

SHOWING THE FINAL REPAYMENT OF THE LOAN, by the operation of the sinking fund, after making the adjustment in the annual instalment, consequent upon a variation in the rate of income upon the present investments to occur at a known future date, without any variation in the rate of accumulation or period of repayment.

		Equivalent amount of original loan.
Amended annual increment for 8 years:—		
Present annual instalment ...	£729·709	
Income from investments, $3\frac{1}{2}$ per cent.	347·648	
	£1077·357	
Amount thereof, accumulated for 8 years at 3 per cent. Calculation (XXVII) 7		£9580·22
Amount of this sum, accumulated for a further 5 years at 3 per cent. Calculation (XXVII) 8		£11106·10
Amended annual increment for 5 years:—		
Annual instalment, as above...	£729·709	
Income from investments, 3 per cent.	297·984	
	£1027·693	
Amount thereof, accumulated for 5 years at 3 per cent. Calculation (XXVII) 9		£5456·16
		£16562·26
Present investments (at end of 12th year)		£9932·74
Amount of original loan		£26495·00

The Rate per cent. **Statement XXVII. C.**

The Amount of (the Amount of £1 per annum).

Method by Step, by Thoman's Tables.

To find the accumulated amount of an annuity to be added to the sinking fund for a limited period of years, and at the end of that period the accumulated amount thereof to continue to accumulate for a further specified period. The rate of accumulation in both cases may be the same, or be at different rates per cent.

Required the amount of an annuity of £347·648 to be added to the sinking fund for a period of 8 years, and accumulated at 3 per cent. At the end of 8 years the annuity ceases, but the sum to which it has then amounted continues to accumulate for a further period of 5 years at 3 per cent.

First period, 8 years; second period, 5 years.

Log. annuity		347·648	2·5411397
add Log. R^N, 3 per cent.	8 years		0·1026978
Log. R^N, 3 per cent.	5 years		0·0641861
			2·7080236
		add 10	12·7080236
deduct Log. a^n, 3 per cent.	8 years		9·1536819
			3·5543417
which is the log. of the required future amount at the end of 13 years			£3583·783

Note. This statement may be compared with Statements XVI. D.1 and XXXIV. G.

Pro forma Sinking Fund Account, No. 12.

A Variation in the Rate of Income from Investments, which is not uniform over the unexpired Repayment Period.

Loan of £26,495, repayable at the end of 25 years.

Showing the final repayment of the loan, by the operation of the increased annual instalment of £729·709.

Statement XXVII. B. Rate of accumulation, 3 per cent.

The amount in the fund at the end of the 12th year, £9932·744, is the correct calculated amount, as shown by Calculation (XV) 2, and by the pro forma account, No. 1, Chapter XV.

Year.	Amount in the fund at beginning of year.	Income received from investments made up to 12th year.	Annual sinking fund instalment.	Income received from investments made after 12th year 3 per cent.	Amount in the fund at end of year.	Year.
1						1
2						2
3						3
4						4
5						5
6						6
7						7
8						8
9						9
10						10
11						11
12					9932·744	12
13	9932·744	347·648	729·709	Nil	11010·101	13
14	11010·101	347·648	729·709	32·320	12119·778	14
15	12119·778	347·648	729·709	65·610	13262·745	15
16	13262·745	347·648	729·709	99·900	14440·002	16
17	14440·002	347·648	729·709	135·216	15652·575	17
18	15652·575	347·648	729·709	171·594	16901·526	18
19	16901·526	347·648	729·709	209·070	18187·953	19
20	18187·953	347·648	729·709	247·654	19512·964	20
21	19512·964	297·984	729·709	287·407	20828·064	21
22	20828·064	297·984	729·709	326·860	22182·617	22
23	22182·617	297·984	729·709	367·494	23577·804	23
24	23577·804	297·984	729·709	409·350	25014·847	24
25	25014·847	297·984	729·709	452·460	26495·000	25

The Rate per cent. Statement XXVII. D.

The Annual Increment (balance of loan) Method.

To find the amended annual sinking fund instalment, consequent upon a known variation in the rate of income upon the present investments to occur at a known future date, based upon the equated annual income.

The rule relating to this method is stated at the head of Chapter XXII.

Amount of original loan (25 years)		£26495·00
deduct amount in the fund at the end of the 12th year		£9932·74
Balance of loan		£16562·26
Annual increment, to be added to the fund, and accumulated at 3 per cent., to provide this amount at the end of 13 years Calculation (XX) 4		£1060·474
deduct the equated annual income to be received from the present investments ascertained as described in the text		£330·765
Amended annual instalment		£729·709
being Present annual instalment ...	£712·826	
Additional instalment, as found in Statement XXVII. A.	£16·883	
		£729·709

Note. This method will be of use where the variation in the rate of income is of the above unequal nature, and is combined with a variation in the rate of accumulation, as in Chapter XXI (Variation C).

Pro forma Sinking Fund Account, No. 13.

A Variation in the Rate of Income from Investments, which is not uniform over the unexpired Repayment Period.

Loan of £26,495, repayable at the end of 25 years.

Showing the final repayment of the loan, by the operation of the increased annual instalment of £729·709, and the equated annual income of £330·765.

Statement XXVII. D. Rate of accumulation, 3 per cent.

The amount in the fund at the end of the 12th year, £9932·744, is the correct calculated amount, as shown by Calculation (XV) 2, and by the pro forma account, No. 1, Chapter XV.

Year.	Amount in the fund at beginning of year	Equated annual income from investments after 12th year.	Annual sinking fund instalment.	Income to be received from investments made after 12th year.	Amount in the fund at end of year.	Year.
1						1
2						2
3						3
4						4
5						5
6						6
7						7
8						8
9						9
10						10
11						11
12					9932·744	12
13	9932·744	330·765	729·709	—	10993·218	13
14	10993·218	330·765	729·709	31·814	12085·506	14
15	12085·506	330·765	729·709	64·583	13210·563	15
16	13210·563	330·765	729·709	98·335	14369·372	16
17	14369·372	330·765	729·709	133·099	15562·945	17
18	15562·945	330·765	729·709	168·906	16792·325	18
19	16792·325	330·765	729·709	205·787	18058·586	19
20	18058·586	330·765	729·709	243·775	19362·835	20
21	19362·835	330·765	729·709	282·903	20706·212	21
22	20706·212	330765	729·709	323·204	22089·890	22
23	22089·890	330·765	729·709	364·714	23515·078	23
24	23515·078	330·765	729·709	407·470	24983·022	24
25	24983·022	330·765	729·709	451·504	26495·000	25

SECTION V.

Sinking Fund Problems. The Date of Borrowing and the Redemption Period

CHAPTER XXVIII.

SINKING FUND PROBLEMS RELATING TO THE DATE OF BORROWING AND THE REDEMPTION PERIOD

WITHOUT ANY COMPLICATION AS REGARDS THE LIFE OR DURATION OF CONTINUING UTILITY OF THE ASSET CREATED OUT OF THE LOAN.

LOAN BORROWED OVER SEVERAL YEARS, IN ONE SUM IN EACH YEAR, EACH YEAR'S BORROWINGS BEING REPAYABLE IN A PRESCRIBED PERIOD FROM THE DATE OF BORROWING.

1. BY MEANS OF ONE SINKING FUND ONLY.
2. BY SEPARATE SINKING FUNDS FOR EACH YEAR'S BORROWINGS.

The foregoing chapters deal with the various problems likely to arise in connection with the sinking funds of local authorities and commercial and financial undertakings affecting (1) the amount in the fund at any time; (2) the rate per cent. of accumulation; (3) the rate per cent. of income upon the present investments representing the fund; (4) the period of repayment; and (5) various combinations of the above factors. In the whole of the examples which have been used to illustrate such problems it has been assumed, for the purpose of calculating the original or amended annual instalment to be set aside and accumulated as a sinking fund to provide a given loan at the end of any period, that the loan was borrowed in *one year and on one date,* namely, at the beginning of the financial year, and that the first annual instalment was set aside at the end of that year. This method of treating an annuity or other periodic payment is the basis of all such calculations and upon which the formulæ and tables are constructed. This ideal procedure may, it is true, be met with occasionally; but as a matter of fact it very seldom occurs in actual practice. It has been assumed in all cases that it has so happened in order to simplify the conditions and to demonstrate the actuarial principles underlying the repayment of debt in this manner, without introducing any extraneous complications. The time has now arrived when it is necessary to consider the conditions occurring in actual practice.

The variations from the ideal method of borrowing are of a twofold nature and arise when the borrowings are made at

various dates in any one year or are spread over several years. If the total loan be repayable on a given date both these variations may necessitate an adjustment in the annual instalment. In the case of a loan borrowed at various dates during any one year the necessity for any adjustment depends upon the magnitude of the loan and affects only the first and last years of the term. Any neglect to make such adjustment cannot prolong the period of repayment for more than part of a year, but in the case of borrowings spread over several years the matter becomes more important. There are other factors which may further complicate the problem, depending upon the nature of the outlay and the periods of repayment allowed for each class. In some cases the power or sanction specifies not only the total amount of loan authorised, but also gives details of the component parts of such loan divided as between the various classes of outlay, each having its own period of repayment. In some cases, however, the total amount authorised is stated without any such subdivision, and an equated period is prescribed for the repayment of the total loan.

The after consideration of the subject will be divided into two parts depending upon the character of the outlay, which may be all of one nature having a similar life or period of utility and a consequent equal period of repayment. On the other hand, the outlay under one power or sanction may consist of various classes for each of which a separate, and varying, period of repayment is imposed.

It may be accepted as a general rule that the repayment period now allowed by Parliament is fixed with regard to the probable life or duration of continuing utility of the works. This variation in the life of the asset imports special difficulties into the problem, relating to the vexed question of the adequacy or otherwise of the sinking fund instalment as a provision for depreciation, obsolescence and supersession.

Dealing first with the actual borrowings, the subject will be treated in the following order, namely:—

I. *Loans authorised for outlay all of one nature having the same period of repayment.*

As regards the actual borrowing such loans may be divided into three classes as follows:

(*a*) Loan borrowed over several years, in one sum in each year, repayable over a term of years in a prescribed period from the several dates of borrowing. Such loans will be described in this chapter.

borrowings as a separate loan repayable in the prescribed number of years from the actual date of borrowing and requiring a separate sinking fund. In the case of a large municipality this entails the keeping of a great number of sinking fund accounts, and the annual provision of the instalments becomes a very detailed process. Further, the final repayment of the loans is spread over a term of years equal to the extended period of borrowing.

In the case of loans raised by the issue of stock the whole of such stock is made redeemable at the end of a specified number of years, or, as is generally the case, on a definite date. In this case there need only be one sinking fund with four separate annual instalments set aside each year for decreasing periods, but all calculated to mature at the same date and at the same rate of accumulation.

The present example will be illustrated by the sinking fund fully described in Chapter XV, relating to the repayment of a loan of £26,495 in 25 years at an accumulation rate of $3\frac{1}{2}$ per cent. In this case it has been ascertained that if the whole of the loan were borrowed in one year, namely, at the beginning of the financial year, it required an annual instalment of £680·234 to be set aside at the end of the first and 24 subsequent years in order to repay the original loan. If the loan, instead of being borrowed in one year, were borrowed over a period of four years, and the sanction or authorisation imposed the period of redemption of 25 years in respect of each amount borrowed, the conditions would have been considerably modified. There is in this case the equivalent of four separate loans each repayable in 25 years, but maturing at the end of four successive years. It will be assumed that each amount of loan was borrowed at the beginning of the financial year, or if borrowed on several dates in that year that no necessity exists to equate the borrowing at the various dates. It will be further assumed that the above amount of £26,495 was borrowed in unequal amounts in each of the four years, and, in order to avoid making four separate additional calculations, that a definite proportion of the loan was borrowed in each year. Seeing that the annual instalment is based upon the amount of £1 per annum for 25 years, it is obvious that it is directly proportionate to the amount of the loan and that the four annual instalments may be found by dividing the original annual instalment of £680·234 in the same proportions as the total loan is divided.

The following table shows the actual details of the loan under consideration:—

TABLE XXVIII. A.

Loan of £26,495, borrowed over four years. Repayment spread over a similar period.

Original annual instalments all calculated to mature in 25 years but at the end of successive years.

Rate of accumulation $3\frac{1}{2}$ per cent.

Year of borrowing.	Redemption period.	Proportion borrowed each year.	Amount borrowed each year.	Annual instalment on yearly borrowing.	Annual instalment at end of each year.
First	25 years	$^2/_{14}$	3785·	97·176	97·176
Second	,,	$^3/_{14}$	5677·5	145·764	242·940
Third	,,	$^4/_{14}$	7570·	194·353	437·293
Fourth	,,	$^5/_{14}$	9462·5	242·941	680·234
			26495·	680·234	—

There are two alternative methods of keeping the sinking fund accounts in such a case. One method is to keep one sinking fund only, and to set aside an increasing instalment during the first four years, a constant instalment during the next 21 years, and a decreasing instalment during the final three years of the total period of 28 years during which the fund will run. If this method be applied to the foregoing example, the annual instalments added to the fund will be as follows:—

TABLE XXVIII. B.

Loan of £26,495, borrowed over four years. Repayment spread over a similar period.

Annual instalments to be added to one sinking fund relating to the total loan.

To be set aside at end of	Annual instalments.		Total.
1st year	97·176	...	97·176
2nd year	242·940	...	242·940
3rd year	437·293	...	437·293
4th year	680·234	...	680·234
5th to 25th year=21 years ...	680·234	...	14,284·914
26th year	583·058	...	583·058
27th year	437·294	...	437·294
28th year	242·941	...	242·941

being the equivalent of 25 annual instalments of £680·234 ... ... £17,005·850

There are several objections to keeping the sinking fund accounts in this manner, all of which are practical. The first is that unless a proper pro forma account be at once made out showing the operation of the fund until maturity there will be a liability to continue the full instalment of £680·234 beyond the 25th year. A further error may possibly arise owing to the application, during the period, of part of the fund in redeeming part of the debt. As previously stated, if any part of the fund be so applied it is requisite and obligatory to pay into the fund, annually, interest upon the loan so redeemed at a rate per cent. at least equal to the calculated rate of accumulation. Although this obligation may be remembered and carried out during the first 25 years it may then be overlooked that at that time £3,785 of loan has been fully redeemed and that interest upon this amount of loan repaid need no longer be charged to the revenue or rate account and added to the fund. The same factor of error may arise at the end of the 26th, 27th and 28th years. Taking the above possible sources of error into consideration, it is preferable to adopt a method which will avoid them, although it may entail a little more clerical work. The proper method in such cases is to keep a separate sinking fund for each year's borrowings, and to prepare at the dates of borrowing a pro forma account showing the operation of each fund until maturity.

It cannot be too often repeated that this pro forma account should be prepared in respect of every sinking fund. If the method of separate sinking funds be adopted it will ensure that proper payments of interest in respect of debt redeemed out of the fund are made to the fund each year and will also enable arrangements to be made to repay each loan at the end of the prescribed period. It will also, in the case of long repayment periods, avoid the necessity of referring to old ledgers or books of account which may have been destroyed.

For this purpose it is an advantage to earmark each fund, and also the corresponding instalment, in some such way as the following:—

Gas Works Sinking Fund. Sanction 1900.
Loan of 1901—25 Years,

and to number each instalment.

The charge to the revenue or rate account at the end of the third year would be made up as follows:—

Sanction 1900—25 Years.

Loan of 1901.	3rd Instalment	... ...	£ 97·176
1902.	2nd do.	... ...	145·764
1903.	1st do.	... ...	194·353
	Total Instalment	... ...	£437·293

At the end of each of the last four years one of the original year's borrowings will be repaid; and the charge to revenue or rate at the end of the 26th year (when the loan borrowed during the first year has been entirely repaid) will be as follows:—

Sanction 1900—25 Years.

Loan of 1901.	Repaid	... ...	nil
1902.	25th Instalment	... ...	£145·764
1903.	24th do.	... ...	194·353
1904.	23rd do.	... ...	242·941
	Total Instalment	... ...	£583·058

Although the method of keeping separate sinking funds for each year's borrowings under each sanction or authorisation is here advocated, it must not be assumed that this method requires that separate bank accounts should be kept for each fund. This would become intolerable in practice even if the bank would agree to do so. Neither is it necessary to keep a separate investment account for each fund. One bank account and one investment account for each department of the local authority is quite sufficient because at the end of any year the amount in the bank, the amount invested, and the loans repaid out of sinking fund should together be equal to the amount standing to the credit of the fund, or to the credit of the whole of the funds of the particular department. In this connection it is important to point out that loans repaid by means of the sinking fund should be treated as an investment of so much of the fund so applied and be debited to a special account instead of being debited to the sinking fund account, in the same way that investments in outside securities are kept in separate accounts. The reason for doing this is to ensure that the revenue or rate account is annually debited, and the sinking fund credited, with the proper amount of interest in respect of such part of the fund so applied in redemption of debt. If the

accounts are kept in such a manner the sinking fund will, at the end of each year, show the amount of loan provided for out of revenue or rate, and will enable a comparison to be made with the pro forma account already referred to and recommended. By this means only can any variation from the calculated amount which should be in the fund at any time be readily ascertained and immediately adjusted. This applies to all sinking funds.

It ought, however, to be pointed out that if only one bank account and one investment account be kept it will be necessary to apportion, as between the different sinking funds, the interest allowed by the bankers and the income received from investments whether the investments be in outside securities or consist of loans redeemed out of the sinking fund. In ordinary cases there may be some difficulty in doing this because the interest allowed by the bank upon balances in hand will almost certainly be at a lower rate than the calculated rate of accumulation of the fund. This difficulty is, however, removed by the fact that there is in the case of each sinking fund a standard to work to, namely, the pro forma account previously prepared showing the amount which should stand to the credit of each fund at the end of each year of the repayment period. If the amount of income actually received from the investment of the sinking fund in outside securities and in loans redeemed falls short of the amount originally calculated to be received, such deficiency should be made good each year by charging it to the revenue or rate account and paying the deficiency into the sinking fund bank account. The necessity to apportion the interest allowed by the bank and the income received from investments may be entirely removed by crediting the interest allowed by the bank, as well as the income received from the investments, to a sinking fund interest suspense account. The suspense account should be debited with the total amount of interest which ought, according to the pro forma accounts, to be credited to the various sinking funds, and the balance remaining to the debit of the suspense account, will show the amount of the deficiency of interest to be debited to the revenue or rate account. By this means not only will the amount standing to the credit of the sinking fund agree each year with the amount which should so stand according to the pro forma account, but there will be the further advantage that each year's revenue or rate account will bear its proper burden and there will never arise any necessity to make provision for a large deficiency in any sinking fund caused by an accumulation of

many annual deficiencies in the income which ought to have accrued to the fund.

This method of keeping separate sinking funds for each year's borrowings is not required in the case of loans issued by way of a stock redeemable at a fixed date, seeing that the repayment of the loan is not spread over a number of years equal to the number of years occupied by the borrowing.

A loan borrowed over a series of years repayable in one sum at a fixed date will be considered in the next chapter, and, for the sake of comparison, the figures used in this example will be further utilised.

CHAPTER XXIX.

SINKING FUND PROBLEMS, RELATING TO THE DATE OF BORROWING AND THE REDEMPTION PERIOD

WITHOUT ANY COMPLICATION AS REGARDS THE LIFE OR DURATION OF CONTINUING UTILITY OF THE ASSET CREATED OUT OF THE LOAN (*continued*).

LOAN BORROWED OVER SEVERAL YEARS, IN ONE SUM IN EACH YEAR, REPAYABLE IN ONE SUM ON A CERTAIN SPECIFIED DATE:

1. WHERE THE DATE OF REPAYMENT IS KNOWN AT THE TIME THE MONEY IS BORROWED.
2. WHERE THE DATE OF REPAYMENT IS FIXED AFTER THE SINKING FUND HAS BEEN IN OPERATION FOR A NUMBER OF YEARS, AND AN ADJUSTMENT OF THE FUND IS REQUIRED.

Summary of the methods of adjustment.

(1) SUMMARY OF THE METHOD *of ascertaining*, at the end of the period of construction, *the future or amended equal annual instalment to be set aside and added to the amount now in the fund which has been provided by the accumulation, during the period of construction, of* temporary instalments *set aside in respect of amounts borrowed over a series of years*, all of which were, and still are, repayable in one sum on a certain specified date which was known at the time the money was borrowed. *The problem is not complicated by any variation in the period of repayment due to the life of the asset.*

(*1*) *Ascertain from the actual records the amount standing to the credit of the fund, at the time the adjustment is required to be made.* *Statement XXIX B.*

(*2*) *Calculate the amount of loan which will be provided at the end of the unexpired repayment period, by the accumulation of this amount now in the fund, at the future rate of accumulation.*
Standard Calculation Form, No. 1.

(3) *Deduct the amount so found, as in (2), from the amount of the original loan.*

(4) *The remainder will represent the balance of loan to be provided by the accumulation, at the future rate, of the required amended annual instalment to be added to the fund during the unexpired repayment period.*

(5) *Calculate the annual instalment so required.*
Standard Calculation Form, No. 3x.

(6) *The annual instalment so ascertained should be equal to the sum of the several annual instalments already set aside in respect of the amounts borrowed in each year provided there is not any variation in the period of repayment or rate of accumulation.* *Table XXIX A.*

(7) *Any variation between the annual instalment so ascertained, as in (5), and the sum of the several annual instalments already set aside will be due to an abnormal past accumulation of the fund, and will result in a surplus or deficiency in the amount of loan to be provided at the end of the repayment period.*

(8) *Such surplus or deficiency (if any) in the amount of loan to be ultimately provided should be corrected in the manner already described under these heads in previous chapters.*

(9) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.* *Statement XXIX. B.*

(10) *Prepare the usual pro forma account.*

(2) SUMMARY OF THE METHOD *of ascertaining*, at some future time, *the future or amended equal annual instalments to be set aside and added to the amount now in the fund which has been provided by the accumulation of previous instalments set aside in respect of loans borrowed over a series of years*, all of which were originally repayable at the end of successive years, but which are now repayable in one sum on a certain date now for the first time specified. *The problem is not complicated by any variation in the period of repayment due to the life of the asset.*

(1) *Ascertain from the actual records the amount standing to the credit of the fund at the time the adjustment is required to be made.* *Table XXIX D.*

(2) *Calculate the amount of loan which will be provided at the end of the unexpired repayment period by the accumulation of this amount now in the fund at the future rate of accumulation.*

Standard Calculation Form, No. 1.

(3) *Deduct the amount so found, as in* (2), *from the amount of the original loan.*

(4) *The remainder will represent the balance of loan to be provided by the accumulation at the future rate of the required amended equal annual instalment to be added to the fund during the unexpired repayment period in substitution for the annual instalment, as originally set aside.*

(5) *Calculate the annual instalment so required.*

Standard Calculation Form, No. 3x.

(6) *Prepare a statement showing the final repayment of the loan by the operation of the sinking fund under the amended conditions.*

(7) *Prepare the usual pro forma account.*

The loans about to be considered differ from the preceding example only in the fact that, although the borrowings are spread over a series of years, the loan is repayable in one sum instead of at the end of successive years corresponding to the number of years during which the money was borrowed. The enquiry is still limited to loans in respect of outlay having a uniform period of repayment. The date of repayment of the loan is generally prescribed in the original sanction or authorisation, and may be either (1) a specified date, (2) a definite number of years from the date of the sanction, or from the commencement of operations, or (3) a given number of years from a date later than the sanction; or, in other words, a deferred sinking fund. On the other hand, the date of repayment may be fixed by the local authority or by Parliament some years after the loan has been borrowed and a sinking fund or funds established. This may arise on the consolidation of existing loans, and also under the following or other similar conditions. A local authority has obtained powers to construct certain works and to borrow on loan, and the power or sanction provides that the loan shall be repaid in 25 years from the dates of borrowing. The actual construction of the works extends over a period of three years, and such an amount only

is borrowed in each year as will pay for the works actually constructed in that year. At the end of three years the works authorised are completed and the full amount of the loan has been borrowed. During the period of construction the proper instalments have been regularly set aside out of revenue or rate, to provide the amount of loan repayable at the end of each of the prescribed periods of 25 years. The local authority then decide to convert the loans into stock redeemable on a fixed date.

This date may be specified under further powers granted, or may be fixed by the local authority at the time of making the adjustment.

The above instances may be divided into two classes requiring different treatment, although the loan relates to outlay of one character only, namely:—

Class 1. Loans in respect of which the date of repayment is known at the time the money is borrowed, and

Class 2. Loans in respect of which the date of repayment is fixed after the sinking fund has been in operation for some years, and an adjustment becomes necessary.

The method of making the adjustment, however, rather than the cause of the adjustment, is the principal object of enquiry.

Class I. Loans in respect of one class of outlay only, borrowed over a series of years, repayable in one sum on a specified date, which date is known at the time the money is borrowed.

The first example will relate to a loan raised by the issue of stock repayable on a specified date. The actual borrowing is spread over three years (the period of construction of tne works), and the period of repayment is 25 years from the commencement of operations. It will be assumed for the purpose of simplifying the conditions, that the local authority has borrowed the money immediately prior to the beginning of the financial year and that work has been commenced on that date. Subsequent borrowings are made on the first day of the two following financial years, and there is not therefore any complication due to the loan being borrowed at various dates in any one year.

The rate of accumulation of the sinking fund is $3\frac{1}{2}$ per cent.

Although it is requisite to keep only one sinking fund, separate calculations must be made of the annual instalments to be charged to revenue or rate, and added to the sinking fund, in respect of each annual amount of loan borrowed. The complete conditions are shown in the following table:—

TABLE XXIX. A.

Loan of £11,355, borrowed over three years, repayable in one sum on a specified date.

Annual instalments calculated for varying periods, all to mature on the same date. Rate of accumulation 3½ per cent.

Year of borrowing.	Redemption period.	Amount borrowed each year.	Annual instalment on yearly borrowing.	Annual instalment at end of each year.
First.	25 years	3,785	97·176	97·176
Second.	24 years	3,785	103·227	200·403
Third.	23 years	3,785	109·836	310·239
		11,355	310·239	

It will be noticed that the loan is borrowed in equal annual amounts during the period of construction, and that the annual instalments in respect of the several amounts borrowed are gradually increased owing to the reduction in the period of repayment. The instalment to be set aside at the end of the first year is £97·176 only and increases until the end of the third year when it attains the maximum of £310·239, which will be continued for a further 22 years when the amount in the sinking fund should be £11,355, provided care has been taken, at the end of each year, to see that the fund has accumulated at the proper rate in accordance with the pro forma account which should have been prepared.

In this instance there is not any decreasing instalment during the later years of the repayment period as was the case in the previous example, Table XXVIII. B, seeing that although the borrowing is spread over three years the instalments are calculated on the basis that the whole of the loan will mature on the same date. The final repayment of the loan is shown in the following statement:—

STATEMENT XXIX. B.

Loan of £11,355, borrowed over three years, repayable in one sum on a specified date.

Showing the final repayment of the loan by the operation of the sinking fund and the annual instalments shown in Table XXIX. A.

AMOUNT IN THE FUND:—		
At end of first year, instalment		£97·176
At end of second year:—		
Interest, $3\frac{1}{2}$ per cent. ...	£3·401	
Instalment	£200·403	
		£203·804
		£300·980
At end of third year:—		
Interest, $3\frac{1}{2}$ per cent. ...	£10·534	
Instalment	£310·239	
		£320·773
Amount in the fund at the end of the third year ...		£621·753
At the end of the 25th year the amount in the fund will be as follows:—		
Amount of £621·753 for 22 years at $3\frac{1}{2}$ per cent. per annum. *Standard Calculation Form, No. 1*		£1325·3
Amount of £310·239 per annum for 22 years at $3\frac{1}{2}$ per cent. per annum *Standard Calculation Form, No. 3*		£10029·7
Total amount of loan		£11355·0

CLASS 2. Loans in respect of one class of outlay only, borrowed over a series of years repayable in one sum on a specified date, such date of repayment being fixed after the sinking fund has been in operation for a number of years and an adjustment is required.

The second class of loans borrowed over a series of years will now be considered, namely, those in which the date of repayment of the whole of the loan is fixed after the sinking fund has been in operation for some years, prior to which time

each year's borrowings were repayable at the end of successive years. In such a case it is necessary to make an adjustment in order to ascertain the future annual instalment to be added to the sinking fund during the whole of the newly ascertained redemption period, in substitution for the varying instalments, as shown in Table XXVIII. B. This adjustment depends upon two factors, namely, the amount now in the fund, and the exact date fixed for the redemption of the whole of the loan. Seeing that the loan in Table XXVIII. B. was originally repayable over a series of four years, and is now repayable on one uniform date, it is advisable to adhere as closely as possible to the original conditions as to repayment, by expediting the repayment of part of the loan and delaying the repayment of an equivalent part. In fact there is here a mild form of the equation of the period of repayment, and an average equation of two years will be adopted since the present example is chosen to illustrate the method of making the adjustment rather than to demonstrate the proper mathematical method of finding the equated period of repayment. This will be fully considered in Chapter XXXII, where it will be shown that the ordinary arithmetical method of finding the equated period is incorrect, but not to such an extent as to make any appreciable difference in two years seeing that in such calculations the nearest whole number of years is adopted.

In the following example the original conditions as to the amounts of loan borrowed, and the annual instalments required, are the same as in Chapter XXVIII, the only difference being that the loan is repayable in one sum instead of at the end of four successive years. The following table contains the original conditions in the example now under consideration, and, as regards the actual figures, is a copy of Table XXVIII. A.

TABLE XXIX. C.

Loan of £26,495, borrowed over four years, repayable in one sum on a specified date fixed after the fund has been in operation for a number of years and an adjustment is required.

Original annual instalments all calculated to mature in 25 years, but at the end of successive years. Rate of accumulation $3\frac{1}{2}$ per cent.

This table is a copy of Table XXVIII A.

Year of borrowing.	Redemption period.	Proportion borrowed each year.	Amount borrowed each year.	Annual instalment on yearly borrowing.	Annual instalment at end of each year
First.	25 years	$^2/_{14}$	3785·	97·176	97·176
Second.	25 years	$^3/_{14}$	5677·5	145·764	242·940
Third.	25 years	$^4/_{14}$	7570·	194·353	437·293
Fourth.	25 years	$^5/_{14}$	9462·5	242·941	680·234
			26495	680·234	

Tables XXVIII. A, and XXVIII. B show the annual instalments to be set aside to repay the above loans at the end of the 25th, 26th, 27th and 28th years.

As stated in the preliminary remarks in this chapter, during the 5th year circumstances arise which render it necessary to provide for the repayment of the whole of the loan on one date, instead of at the end of 4 successive years, and it will be assumed that the end of the 26th year is adopted as the redemption date. Four separate sinking funds have been kept and each fund stands at the proper amount shown by the pro forma account. This means that the accumulation of each fund by way of income from investments has been equal to the calculated amount, or that any deficiency has been made good year by year out of revenue or rate.

In case there is a deficiency or a surplus in the fund at the time of making the adjustment it may be accurately adjusted if necessary by the methods fully described in previous chapters, but as a general rule unless the discrepancy is of large amount it is merged in the general adjustment about to be made. In ordinary practice of course the present position of the fund is ascertained from the actual records or books of account, but in the present example the amount in the fund must be found by actual calculation.

The first step therefore is to ascertain the amounts which should stand to the credit of each of the individual sinking funds relating to each of the four year's borrowings at the end of the fourth year, this being the date when the maximum instalment has been set aside in respect of the full amount of the loan which has then been borrowed.

This may be done by the following arithmetical calculation which is somewhat shorter than by the tables and logarithms and which has the further advantage that it shows, although in decimal form, the actual entries in the ledger.

TABLE XXIX. D.

Loan of £26,495, borrowed over four years, repayable in one sum on a specified date fixed after the fund has been in operation for a number of years, and an adjustment is required.

Separate sinking funds. The amount in each fund at the end of the fourth year, will be as follows:—

Amount set aside at the end of the following financial years.		Sinking Funds in respect of loan borrowed at the beginning of the following financial years. First.	Second.	Third.	Fourth.
First	Instalment	97·176			
Second	Interest	3·401			
	Instalment	97·176	145·764		
		197·753	145·764		
Third	Interest	6·921	5·102		
	Instalment	97·176	145·764	194·353	
		301·850	296·630	194·353	
Fourth	Interest	10·565	10·382	6·802	
	Instalment	97·176	145·764	194·353	242·941
		409·591	452·776	395·508	242·941
					395·508
					452·776
					409·591

Amount standing to the credit of the four sinking funds at the end of the fourth year of borrowing ... ... £1500·816

The accuracy of the above calculation may be proved by assuming that only one sinking fund had been kept. Although this is not recommended in the case of a loan repayable over a series of years, there is an advantage in keeping only one fund where the loan is repayable in one sum provided a pro forma account of the operation of the fund is prepared when the full amount of loan has been borrowed. The following table shows the amount in the sinking fund at the end of the fourth year:

TABLE XXIX. E.

Loan of £26,495, as above.

One sinking fund only. The amount in the fund at the end of the fourth year, will be as follows:—

Amount added to the fund at the end of the following years.	Amount to credit of fund at beginning of year	Annual accretions. Interest $3\frac{1}{2}$ %	Instalment.	Amount to credit of fund at end of year.
First ...			97·176	97·176
Second ...	97·176	3·401	242·940	343·517
Third ...	343·517	12·023	437·293	792·833
Fourth ...	792·833	27·749	680·234	1500·816
		43·173	1457·643	1500·816

At the time of making the adjustment four sinking funds are in operation, all relating to the repayment of a loan of £26,495 borrowed in unequal amounts over a period of four years, and each year's borrowings are repayable in 25 years from the date of the original borrowing, the last portion of the loan being repayable at the end of the 28th year. The actual repayment of the total loan was originally spread over a period of 4 years, but under the new conditions it is required to amend the annual instalment of £680·234 to be set aside for 21 years followed by decreasing instalments for 3 years as shown in Table XXVIII. B. In place of these varying instalments required to repay the loan at the end of four successive years it is necessary to ascertain the annual instalment which will repay the whole of the loan of £26,495 at the end of 22 years from the present time, bearing in mind that there is in the fund an amount of £1500·816 which can be applied in reduction of the future annual instalment.

The amended annual sinking fund instalment may be found in the following manner which is similar in principle to the annual increment (balance of loan) method described in Chapter XXII:—

Amount of loan, repayable in 22 years from the present time	£26495·00
Deduct therefrom the amount of loan which will be provided by the accumulation at $3\frac{1}{2}$ per cent. for 22 years of the £1500·816 now in the fund. By standard calculation form No. 1...	£3199·07
leaving a balance of loan of	£23295·93

to be provided by the accumulation at $3\frac{1}{2}$ per cent. of the future amended annual sinking fund instalment to be set aside for 22 years.

This amended annual instalment as may be found by standard calculation form No. 3 x, is £720·59.

Proof of the above Adjustment. In ordinary practice, of course, the best method of proving the above calculation is to prepare the usual pro forma account so often recommended, showing the amount which should be in the fund at the end of each year, and which is required in order to control the subsequent accumulation of the fund. This method, however, is unsuited to a work of this nature, and it is preferable to adopt a method of proof based upon actuarial principles.

To recapitulate the data. A loan of £26,495 is repayable at the end of 22 years, and there is in the sinking fund the sum of £1500·816 which will accumulate at $3\frac{1}{2}$ per cent. The problem is to ascertain the sinking fund instalment to be set aside and accumulated for the remaining 22 years.

The calculation is made in two stages as follows:—First ascertain the annual instalment to be set aside and accumulated as a sinking fund at $3\frac{1}{2}$ per cent. to provide £26,495 at the end of 22 years. This annual instalment is £819·54.

The next step is to ascertain the annual sum or annuity by which this instalment will be reduced by the amount of £1500·816 now in the fund, or in other words the annuity for 22 years at $3\frac{1}{2}$ per cent. which may be purchased by the above sum of £1500·816. This annual amount, using the author's standard calculation form No. 5, will be found to be £98·95.

The adjusted annual instalment therefore is:—

Annual instalment to repay the loan of £26,495 in 22 years	£819·54
less the reduction therein due to the amount of £1500·816 now in the fund	£98·95
Amended annual instalment as previously ascertained	£720·59

In the foregoing example it has been assumed that the amount of the loan remains unchanged and that the rate of accumulation of the fund and the income from investments will continue to be $3\frac{1}{2}$ per cent. as in the original example in Chapter XV. The only variation is in the period of repayment. The methods described in Chapter XXIV, variation in the period of repayment, cannot be adopted because the amended annual instalment here required is to replace four instalments to be set aside for varying periods instead of one instalment for one period. The method will apply equally to loans not raised by the issue of stock if the whole of the loans are repayable in one sum on a specified date.

CHAPTER XXX.

SINKING FUND PROBLEMS, RELATING TO THE DATE OF BORROWING AND THE REDEMPTION PERIOD,

WITHOUT ANY COMPLICATION AS REGARDS THE LIFE OR DURATION OF CONTINUING UTILITY OF THE ASSET CREATED OUT OF THE LOAN (*continued*).

LOAN BORROWED IN ONE OR MORE YEARS IN VARYING AMOUNTS AT VARIOUS DATES IN EACH YEAR, AND IT IS REQUIRED THAT THE REVENUE OR RATE ACCOUNT OF EACH YEAR SHALL BE CHARGED WITH A PROPORTIONATE PART OF THE ANNUAL SINKING FUND INSTALMENT.

The actual borrowings *in any one year* (whether in respect of a loan borrowed entirely in one year, or borrowed over a series of years depending upon the period of construction) are often made at various dates during the year because the money is not required or is not readily obtainable. To carry out the strict letter of the obligation to repay the loan at the end of a prescribed number of years from the date of borrowing would be practically impossible if each individual borrowing had to be treated separately. The general practice is to treat all the sums received in any one year as if they had been borrowed at the end of the financial year and not to set aside any sinking fund instalment in respect of the broken period of the year of borrowing, the provision of the first full annual instalment being deferred until the end of the succeeding financial year, which simplifies the working of the fund very considerably. In the case of a small loan borrowed piecemeal in this fashion in one year there is not any great objection to outweigh the manifest advantages; and the same applies to loans borrowed over a period of years during construction in which the annual amount borrowed is not large. The principle of deferring the first annual contribution has been extended by Parliament in certain cases, where the operation of the sinking fund has been suspended for a specified number of years

But it may happen in the case of a local authority or a commercial or financial undertaking that the loan is of large amount and may be borrowed during one year. It may be necessary and equitable in such a case to charge the revenue or rate account of that year with the proper calculated proportion of one year's annual instalment in respect of each separate borrowing, based upon the part of one year for which the undertaking has had the use of the money raised during the year, and not defer the first annual contribution until the end of the succeeding year.

Such an instance might arise in connection with the purchase of an existing undertaking by a local authority where the purchase money is payable by instalments spread over a year and is borrowed as and when required, but the local authority enters into possession immediately and takes the whole of the profits. If no contribution to the sinking fund were made during the first year the revenue or rate account of that year would show a fictitious profit as compared with subsequent years. In such a case it would appear not only equitable but good accounting practice to charge the revenue or rate account of that year with a portion of the annual instalment commensurate to the amount of loan it has had the use of during part of the year.

In the case of a commercial undertaking the revenue account for the year would of course be charged with interest upon the loan for the exact number of days the money had been borrowed, and the same would apply to the revenue account of a local authority where the accounts are kept upon the "income and expenditure" as distinguished from the "receipts and payments" system.

If the principle applies to interest upon the loan, it should certainly apply to the annual contribution to the sinking fund, especially in the case of a local authority where both amounts are specific charges against revenue or rate. In the case of a commercial undertaking the conditions as to a sinking fund are much more elastic than is the case with the loans of local authorities, and much would depend upon the actual conditions laid down in the deed governing the loan, which would be taken into account by the auditors before certifying the accounts.

In the case of local authorities it is impossible to lay down any hard and fast rule. The conditions imposed upon such authorities have of late years been of a uniform nature depending upon the probable life of the asset, but where powers are granted by special Act of Parliament wider latitude has often

been allowed, and the special nature of the powers requires careful scrutiny in each case. Attention may, however, properly be directed to the magnitude of the loan; in some instances the amount involved may be considerable, and may point to the necessity of making some such adjustment, but to insist upon it in all cases, irrespective of the amount of the loan, might, and possibly would, involve considerable labour without any corresponding advantage.

With regard to the actual adjustment, there are several interesting points, and the problem is not so simple as it appears at first sight. To find the actual proportion of the annual instalment to be charged to the revenue account of the year of borrowing it is first necessary to ascertain the annual instalment to repay the total loan borrowed, having regard to the redemption period imposed. Seeing that an adjustment of this nature is rarely made in the case of small loans, but is confined to loans of considerable magnitude, it is very important that the calculation should be made with extreme accuracy. Such large loans are generally raised by the issue of stock redeemable on a fixed date, and it often happens that the total amount of the loan is borrowed over a period of years, rendering it necessary to make a similar calculation of the proportionate part of one year's annual instalment at the end of each year of borrowing. In this manner varying amounts are added to the fund each year, which departs from the normal growth of a sinking fund by equal annual instalments. This will render it necessary to set aside each year what may be termed temporary instalments, and to adjust the fund when the whole of the loan has been raised, by ascertaining the exact equal annual instalment required to be set aside during the remaining years of the redemption period to repay the loan on the prescribed date, having regard to the amount in the fund at the time of making the adjustment.

The problem will be illustrated by a sinking fund to repay a loan of £11,355 in one sum, on a specified date (namely, at the end of 25 years) with a rate of accumulation of 3½ per cent., the loan being borrowed in three equal annual sums of £3,785. These amounts are borrowed at various dates during the several financial years, and it is required that the revenue or rate account of each year shall be charged with a proportionate part of the sinking fund instalment in respect of the money borrowed during the year.

A similar loan has already been used to illustrate the example in Chapter XXIX, in which case the money was

supposed to be borrowed on the first day of each financial year, and the conditions shown in Table XXIX. A. will be adopted in the present instance in order to show the effect as compared with that example, although the amounts are small. But the principle is the same, and the effect upon a larger loan will be readily appreciated when it is remembered that, given the same number of years and rate of accumulation, the annual instalment is always proportionate to the loan.

A further imaginary factor must be assumed, namely the precise date or dates in each year on which the loan was raised.

It is more than probable that the loan borrowed in any one year will be raised in more than one sum. In such cases it is sufficiently if not quite correct to proceed by the arithmetical method and multiply the several amounts borrowed by the number of days between the respective dates of borrowing and the end of the financial year. The sum of these products divided by the total amount borrowed during the year and the result again divided by 365 is the proportion of the year required.

The following example will make the matter clear:—

TABLE XXX A.

To ascertain the proportion of the annual instalment in respect of the amounts borrowed during one year.

The Arithmetical Method.

Date of borrowing.	Amount borrowed.	Number of days to end of financial year.	Product of amount X days.
January 31	£100	334	33,400
March 31	£200	245	49,000
June 30	£300	184	55,200
September 30	£400	92	36,800
Total	£1000		174,400

The equivalent proportion of one year for which the undertaking has had the benefit of the £1000 is arrived at as follows:—

$$\frac{174,400}{1000} \div 365 = \frac{174 \cdot 4}{365}$$

and this proportion of the annual sinking fund instalment is chargeable against the revenue or rate account of the year of borrowing.

In order, however, to simplify the following calculation it

will be assumed that the loan was raised in each year in one sum, and that the local authority had the use of the money for the following portions of each year:—

First year	one half of the year
Second year	one third of the year.
Third year...	one quarter of the year.

The exact dates of borrowing during each year have a very important effect upon the variation in the annual instalment during the period of borrowing and the subsequent period of repayment. If the amounts are borrowed during the early part of the year, the proportionate part of one year's instalment will be greater than if the money were borrowed during the later part of the year.

The complete conditions are shown in the following table:—

TABLE XXX B.

Loan of £11,355 borrowed over 3 years, repayable in one sum on a specified date, by means of an annual sinking fund instalment to accumulate at $3\frac{1}{2}$ per cent. The revenue or rate account of each year to be charged with a proportionate part of the annual instalment in respect of the amount of loan borrowed during such year.

Annual amounts borrowed and yearly and proportionate instalments.

Year.	Amount borrowed.	Portion of year for which money borrowed.	Period in which repayable	Annual instalment. Yearly.	Annual instalment. Proportionate part of first year's instalment.
First	3785	$\frac{1}{2}$	25 years	97·176	48·588
Second	3785	$\frac{1}{3}$	24 years	103·227	34·409
Third	3785	$\frac{1}{4}$	23 years	109·836	27·459
	£11,355			£310·239	

NOTE.—This table should be compared with Table XXIX. A.

The above annual instalments are calculated for even periods of 25, 24 and 23 years respectively, and in the following example it will be assumed that they are set aside during the three years, at the end of which period the necessary adjustment will be made. This is the most practical way of dealing with the matter, although it may properly be contended that the above yearly instalments should be slightly reduced in consequence of the proportionate parts set aside in respect of the year of borrowing. The main object of the adjustment is to ensure

that the revenue or rate account of the year of borrowing shall be charged with a proper proportion of the sinking fund instalment rather than that subsequent years shall be charged to a fraction with the exact mathematical amount.

The following table shows the complete and proportional instalments which will be added to the fund during the three years, and the amount which should be in the fund at the end of the third year of borrowing. The broken year during which the first amount was borrowed is not included in the period of repayment, which is in effect extended by part of a year.

STATEMENT XXX C.

Loan of £11,355, borrowed over three years, repayable in one sum on a specified date.

A proportion of each annual instalment to be set aside in respect of the amounts borrowed during each year.

The amount in the sinking fund at the end of each year of borrowing, will be as follows:—

Borrowing begins

at beginning of first year of fund:—

$\frac{1}{2}$ of £97·176, instalment, first year...		48·588

Repayment period begins

at end of first year of fund:—

Interest on £48·588	1·700	
Instalment, first year	97·176	
$\frac{1}{3}$ of £103·227, instalment, second year	34·409	
		133·285
		181·873

at end of second year of fund:—

Interest on £181·873	6·366	
Instalment, first year	97·176	
Instalment, second year	103·227	
$\frac{1}{4}$ of £109·836, instalment, third year	27·459	
		234·228
		416·101

BORROWING CEASES

at end of third year of fund:—

Interest on £416·101	14·564	
Instalment, first year	97·176	
Instalment, second year	103·227	
Instalment, third year	109·836	
		324·803
Amount in the fund at the end of the third year ...		£740·904

The above amounts credited to the sinking fund are contributed as follows:—

	Charged to revenue account.	Interest from investments.	Total
First year of borrowing	48·588	—	48·588
First year of repayment period	131·585	1·700	133·285
Second year of repayment period	227·862	6·366	234·228
Third year of repayment period	310·239	14·564	324·803
	718·274	22·630	740·904
as compared with the previous example in Statement XXIX B.:			
First year of repayment period	97·176	—	97·176
Second year of repayment period	200·403	3·401	203·804
Third year of repayment period	310·239	10·534	320·773
	607·818	13·935	621·753
or a surplus of	110·456	8·695	119·151

There is in the fund at the end of the third year the sum of	£740·904
as compared with the previous example, Statement XXIX. B.	£621·753
a surplus of	£119·151

being the accumulation of the proportionate parts of the instalments set aside in respect of the years of borrowing, as may be verified by a similar calculation.

Seeing that the loan is the same in amount and the unexpired period is 22 years in each case this surplus will tend to reduce the annual instalment of £310·239.

The reduced annual instalment may be found in the following manner which is similar in principle to the annual increment (balance of loan) method described in Chapter XXII:—

Amount of loan repayable in 22 years from the present time	£11355·00
Deduct therefrom the amount of loan which will be provided by the accumulation at $3\frac{1}{2}$ per cent. for 22 years of the £740·904 now in the fund. By standard calculation form, No. 1	£1579·24
leaving a balance of loan of	£9775·76

to be provided by the accumulation, at $3\frac{1}{2}$ per cent., of the future amended annual sinking fund instalment to be set aside for 22 years.

This amended annual instalment, as may be found by standard calculation form, No. 3x, is	£302·38

Proof of the above Adjustment. The accuracy of the above adjustment may be proved in a similar manner to that adopted in Chapter XXIX. A loan of £11,355 is repayable at the end of 22 years, towards which there is in the fund an amount of £740·904, which will accumulate at $3\frac{1}{2}$ per cent.

The annual instalment to repay the loan of £11,355 in 22 years at $3\frac{1}{2}$ per cent., as may be found by standard calculation form, No. 3x, is	£351·23
but the amount of £740·904 now in the fund is equivalent to an annual instalment for the same period, as may be found by standard calculation form, No. 5, of	£48·85
leaving a reduced annual instalment, as previously ascertained, of	£302·38

Two methods have now been described of repaying a loan of £11,355 (borrowed over a period of 3 years) at the end of 25 years under two sets of conditions, namely:—

A, where the annual instalment is set aside at the end of the financial year following the year of borrowing, and the revenue or rate account of the year of borrowing is relieved of any charge in respect of the sinking fund instalment. Chapter XXIX, Table XXIX. B.

B, where the revenue or rate account of each year of borrowing is charged with a proportionate part of the annual instalment. Chapter XXX, Table XXX. C.

The annual charges to revenue or rate account in each case may be usefully compared by means of the following table:—

TABLE XXX. D.

Loan of £11,355 borrowed over three years, repayable in one sum on a specified date.

A. Annual instalments only. *Table XXIX. B.*

B. Annual and proportional instalments. *Table XXX. C.*

Comparison of the annual charges to revenue or rate in respect of the sinking fund instalment.

Amount charged to the revenue or rate account.	A. Where the year of borrowing is relieved of any charge in respect of the sinking fund instalment. Table XXIX. B.	B. Where the year of borrowing is charged with a proportionate part of the sinking fund instalment. Table XXX. C.	Excess of B. over A.
First year of borrowing ...	Nil	48·588	48·588
First year of repayment period	97·176	131·585	34·409
Second year of repayment period	200·403	227·862	27·459
Third year of repayment period	310·239	310·239	Nil
Total	607·818	718·274	110·456
each of the subsequent 22 years of the repayment period	310·239	302·380	7·859

The effect of charging the revenue or rate account of the year of borrowing with a proportionate part of the sinking fund instalment instead of deferring any charge to the end of the following financial year may be summarised as follows:—

The charge to revenue or rate account is antedated by one year to the extent of the proportionate instalment in respect of the first year's borrowings, and the same applies to each year during which the borrowing takes place. The difference between the two methods affects only the revenue or rate accounts of the years of extended borrowing, but as the first broken year of borrowing is not included in the repayment period, the annual instalment charged to revenue or rate in the third year of the fund is the same in both methods.

Having thus charged the earlier years with a greater part of the repayment burden, it is obvious that the later years will be correspondingly relieved, and the above table shows this to be the case. But the increased burden to revenue or rate account during the years of borrowing is spread over a smaller number of years than the relief is obtained during the remainder of the repayment period, in consequence of which the effect is to charge the revenue or rate accounts of the years of borrowing with a far greater annual amount than that by which subsequent years are relieved.

In the above example the three years of borrowing are charged with an additional amount of £110·456, or an average of £36·818 per annum, whereas the subsequent years are relieved to the extent of £7·859 per annum only.

The above amounts of additional burden during the earlier years and the corresponding amounts of relief during the later years must not be accepted as an exact ratio which will apply to all examples of this nature, because the dominant varying factor in the foregoing adjustment is the actual date or dates in each year upon which the loan was borrowed. If the loan had been borrowed on the first day of the financial year the two methods would yield exactly similar results, but if the loan had been borrowed during the early part of the previous year the results would have shown much more variation than the average example used to illustrate the subject.

The necessity to make an adjustment of this nature therefore depends, primarily, upon the magnitude of the loan, and, secondly, upon the portion of the year during which the money borrowed has been utilised.

Section VI.

The Life or Duration of Continuing Utility of the Asset Created out of the Loan, and its Relation to the Redemption Period and the Incidence of Taxation

CHAPTER XXXI.

THE LIFE OR DURATION OF CONTINUING UTILITY OF THE ASSET CREATED OUT OF THE LOAN, AND ITS RELATION TO THE REDEMPTION PERIOD AND THE INCIDENCE OF TAXATION.

In the case of the loans of municipal or other local authorities, there is a further factor which requires serious consideration, namely, the periods allowed by Parliament (or the Government Department concerned) for the repayment of loans authorised for different classes of outlay having longer or shorter lives or periods of duration or utility, and this variation in the life of the asset may in its turn react upon the period over which the loan is borrowed or is repayable. This factor gives rise to the necessity to equate the period during which loans shall be repayable depending upon,

1. The life of the asset and the consequent period of repayment.
2. The date or dates of borrowing, whether in one year or spread over a period of years.
3. A combination of both periods, namely, of borrowing or repayment.

This is the most difficult problem in municipal finance, upon which there is much divergence of opinion, as is only natural considering the extended and complicated nature of municipal activity, which, as all who have paid attention to such matters know, is ever widening.

Communities have not any capital beyond the liability of each citizen of both the present and future generations to contribute his rateable proportion of the cost of the benefits which he receives from the joint efforts of the community. Such benefits are received by each citizen in each generation year by year, and should be paid for as and when received. In a primeval community individual benefit is paid for by individual labour, but such an ideal method of contribution can only exist in a small community, and the difficulty of

apportioning the annual burden in a rapidly growing one is intensified in a far greater ratio than the actual numerical increase of population.

Year by year, as the community grows, the problem becomes more complicated. Works of public utility, which in a small community might be ignored or neglected, become of vital importance, and must be carried out, and in doing so regard must be had not only to the present requirements, but also to the future growth. It is obviously useless to undertake public works which it is well known will be utterly inadequate to provide for the needs of future generations, and provision must be made in advance.

This increases the cost of all works of public utility, and involves the immediate spending of large sums of money which cannot be found by, and cannot properly be charged against, the present generation of ratepayers, either at once or spread over comparatively few years. Such outlay can only be met by pledging the credit of the community for the purpose of raising a loan. Consequently the repayment of the loan must be spread over an extended period depending upon—

(1) the probable life of the asset upon which the money is expended;

(2) the liability of future generations to provide further works of public utility which may then be required; and

(3) the judgment of those immediately responsible for the adequacy of the present outlay, including in such term not only the actual permanence of the work undertaken, but also the probability that future advancements in knowledge may render such works either inadequate in design or too costly in operation. This throws the responsibility of the actual outlay upon those who incur it, and it is now a generally accepted principle that the cost of all outlay upon works of public utility should be written off, and the loans raised therefor actually repaid, out of current revenue or rate during a period well within the life of the particular works to provide which the loan is borrowed. It is obvious, therefore, that the provision of public utilities adequate to the needs of future generations in any individual community is far too great a burden to be imposed upon the present generation of ratepayers, and that this involves pledging the future credit of the community. By a parity of reasoning the increase in size and number of communities, and the ever widening sphere of local activities, renders it imperative that the extent to which the present generation shall be allowed to

pledge the credit of the future should be treated not as a local but rather as a national question. At the present time, therefore, all loans raised by local authorities for purposes of public utilities are subject to the final approval of Parliament, but owing to the enormous increase in this direction Parliament has been compelled to delegate its powers as to detail to Committees and to certain Government departments. This has been a very gradual process extending over many years, during which time many Acts have been placed upon the Statute Book, with the result that powers have been obtained under both General and Special Acts, and this has led to considerable difference in practice. The great disadvantage of this variation consists in the fact that the larger municipalities, instead of seeking powers under General Acts, may, in many cases, avoid the careful scrutiny of the permanent Government departments (which now proceed upon regularly defined principles) by applying to Parliament for a Special Act. All such Special Acts are referred to Committees composed of members of both Houses of Parliament, but there is not any continuity in the membership of such Committees, and as the permanent Government departments are not represented thereon, there is not any uniformity of practice, and the result is seen in the extreme variation in the powers as to borrowing and repayment now existing. The present general policy of Parliament and of the Government departments charged with the duty of fixing the respective periods of repayment operates in the direction of equalising the period of repayment and the life of the asset, although the conditions now in force vary considerably in individual cases for the reasons already stated.

This principle is of modern growth. In the early days of municipal government, *i.e.*, prior to 1847, the Acts authorising expenditures upon public utilities did not impose any obligation of any kind to repay the loan out of annual rates to be levied upon the community, and there are to-day many loans outstanding in respect of which no such obligation exists, and the debt and the interest payable thereon may for all practical purposes be considered as a perpetual charge upon the annual rates to be levied by the municipalities unless and until they voluntarily provide for its redemption by making annual charges against revenue or rate. In some cases this provision has been made on the initiation of those responsible for the financial administration of the municipality, and in other cases such delayed provision has been imposed by Parliament as a condition precedent to the granting of further borrowing powers.

There is now, however, a considerable body of municipal opinion that where the money borrowed is expended in the purchase of land in or near the centre of a city, and the erection thereon of buildings of a substantial nature and of assured future utility, the asset may be considered as of permanent and in many cases even of improving value, and that there is not therefore any necessity to burden the ratepayers of the present or any future generation with any charge in respect of the redemption of the debt beyond the annual interest payable upon the loan which interest may, it is contended, properly be considered as the equivalent of an annual rent.

In support of an argument of this nature it is contended that local authorities may, and very often do, occupy lands and premises as ordinary tenants, paying therefor the usual rents demanded by the owners of the property, and such tenancies may be of an annual nature or be by way of lease for a term of years. Such leases for years may be of short duration, but, on the other hand, they may, in certain districts, be for very long terms, possibly longer than would be granted by Parliament for the repayment of a loan authorised for the purchase of the property. In such cases it is obviously to the advantage of the local authority to acquire the property by way of lease rather than by purchase, seeing that there will not be any burden in respect of the sinking fund instalment for the redemption of the loan. Especially does this apply to the acquisition of land or buildings which do not immediately require any large outlay or where the outlay is of such a character that it may be spread over a number of years and be met by charging it direct to current annual revenue or rate, or where the annual outlay may be so arranged that it is less than the sinking fund instalment to be set aside to repay the loan necessary to be raised to purchase the property. Such conditions may not always exist, especially in the case of outlay in respect of land required for purposes of public parks or open spaces, or large public buildings, such as town halls, requiring a large expenditure upon buildings, but the principle is important and may be applied to the occupation of land and buildings without imposing any burden upon the present and future generation of ratepayers for the acquisition of properties which may at any future time be replaced by others, which may be not only as cheaply acquired but may be more suitable for the purpose. As against this it is argued that land in the centre of a city required for the erection of a town hall, or land for public parks, increases rapidly in value, and at the end of a long lease the fine or

premium payable on renewal of the lease would be very large, and the probability of such a burden being laid by the present upon the shoulders of a future generation would certainly not be sanctioned by Parliament.

MARKETS. The argument appears to be equally strong when applied to markets which generally occupy land near the centre of the city and in respect of which the cost of the land is the predominant factor, since the buildings are not usually of an expensive character. In addition to the improving value of the site, markets are a source of revenue consisting of tolls upon produce and rents of floor space and buildings, which revenue, after providing for all charges, yields a surplus which is applied in aid of the rates levied upon the general body of ratepayers. In most cases markets yield a surplus revenue over and above all upkeep charges, and it seems only proper that the present generation of ratepayers should out of such surplus revenue provide an annual instalment towards the redemption of the debt before applying any profits in aid of their annual rateable contributions towards the upkeep of the city.

WATER. The provision of a permanent supply of pure water for sanitary and other purposes is the prime necessity of all communities for many weighty reasons, and demands special consideration. The paramount factor in this case is the imperative obligation to provide for the needs of the community for a number of generations far in excess of that requisite in the case of any other public utility; indeed, it may properly be contended that it is the duty of the present generation to ensure that a permanent supply of pure water sufficient for the needs of the community shall continue for ever. Methods of lighting, transportation, sewage disposal and other communal necessities are being constantly improved, and any future improvements in such comparatively minor utilities may be carried out upon land already allocated to them and acquired by the municipality. But with water supply the conditions are the exact opposite. Owing to the rapid growth of cities involving increasing demands for water for sanitary and manufacturing purposes, the natural areas suitable for the supply of water are being year by year continually encroached upon and reduced, and future improvements in methods of transportation will enable manufacturing processes to be profitably carried on far beyond the present city limits. Such conditions are favourable to the creation of vested interests in all land which is a

natural water area, and such vested interests will be scattered in such a manner as to render their acquisition at some future time practically impossible even at any price. It is therefore the duty of all municipalities to protect and preserve all natural water areas for the public use and to expend money upon the purchase far in advance of present requirements. There is here an obligation to pledge the credit of the community for the purchase of land and the construction of works to provide a sufficient supply of water to meet the maximum needs of the community, and yet the present policy of Parliament is to allow a shorter period than formerly. Owing to the reasons already mentioned such land is continually increasing in value, and many existing water undertakings are now worth very much more than their original cost. It is therefore argued that the repayment of money borrowed to provide the cost of land for water areas should be spread over a very long period of years, even if it be not treated as a debt in perpetuity.

The argument as to the large amount expended in the purchase of land is supported by the substantial and permanent character of the works erected thereon, and it seems at first sight sound policy to relieve the present generation of ratepayers from what appears to be an undue burden by spreading the redemption of the loan over a longer period than is at present allowed by Parliament. As against this it is pointed out that water works have failed, water areas have yielded a decreased and insufficient supply, and works which were once thought adequate have, owing to the large increase of towns, become insufficient and have had to be supplemented by further outlay. It is also contended that if the repayment of the debt be spread over a very extended period the interest paid equals, and soon exceeds, the amount of principal. This is not in itself a very good reason against extended periods of repayment seeing that its effect is to spread the burden over a greater number of generations who derive benefit from the outlay, provided always that the works continue to meet the needs of the community and subsequent generations do actually derive a benefit therefrom.

But it is common knowledge that very few works of public utility last for more than a certain number of years. In some cases the rateable value of a district falls, but in nearly all cases the future demands of the community increase so rapidly that it is imperative to put what may by some be termed an undue burden upon the present generation in order to avoid placing an intolerable burden upon the future. The personal

element also enters largely into the matter, and it has been found that the surest, if not the only, way to check undue expenditure, if not extravagance, upon the part of local authorities is to convert each £1,000 of capital outlay into a definite proportion of the annual amount payable by the ratepayer by way of rate, and, further, to educate the ratepayer to appreciate this. There is also another interest to be considered, namely, the loanholder who finds the money and who, in a great majority of cases, has not any local interests. He looks solely to his security both for the annual payment of interest and the ultimate repayment of his capital. His security consists partly of the assets created out of his money and partly of the annual revenues derived therefrom, but in practice mainly of the future annual rates to be levied upon the community. Seeing that the value of the communal assets depends entirely upon the perpetual prosperous existence of the community, such assets have really no value unless the community is able to pay the future annual rates. A bankrupt or insolvent community, if not an absolute impossibility, would not be able to pay any serious percentage of its liabilities; and seeing that the security for its loan indebtedness is a mixed fund of capital and revenue, of which the latter is the chief, it seems not only reasonable, but just, that revenue or rate should bear the greater proportion of the burden. It follows, therefore, that the cost of the outlay should be repaid within the productive life of the asset and be charged against the annual rates levied by the local authority. In the case of revenue earning undertakings it may, not very unreasonably, be contended that any surplus profits should partially, if not wholly, be applied in redemption of debt instead of in aid of rate. If the whole of such profits were applied in redemption of debt, it would avoid the present anomaly of towns with equal annual rates but with widely varying expenditures, due solely to the fact that the excess expenditure in one case is concealed by the profit derived from trading departments. In the case of tramways this profit is fairly earned since there is a generally accepted level of fares all over the country, but in the case of gas and electric lighting undertakings there is such a wide divergence of charges as between different municipalities, that a very high charge, levied at will by the local authority, is called a profit, and is taken out of the pockets of one class of ratepayers, namely, the gas or electricity consumers, and applied in relief of the rates paid by the whole community.

The foregoing remarks deal very fully with waterworks as representing a class of outlay which lends itself most readily to the argument in favour of a total abandonment of the annual charge against revenue or rate in respect of the redemption of debt, or at least in favour of a reduction in the annual charge to the present generation of ratepayers to the possible and probable detriment of future generations. They will have their own burdens to bear both as to their then present, and future obligations. Any relaxation of the present, as some think, stringent regulations and practice will most probably give them in addition a past burden to bear, which, owing to the foresight of our local authority forefathers we have escaped. Consequently the modern Parliamentary practice is right, namely, to require the redemption of the loan to be spread over a period well within the life of the asset created out of the loan and to differentiate between various classes of outlay in fixing the period to be allowed in respect of each.

Outlay on Manufacturing Plants. So far the enquiry has been confined to capital outlay upon public works in which the greater part of the cost is for land, which rarely depreciates, and very often appreciates in value, or for buildings for which a very long life may reasonably be expected seeing that judicious outlay upon repairs will prolong the life considerably. There is, however, a further class of outlay of a much more complex nature where the proportion of the original cost attributable to land is comparatively small, and the greater part of the outlay is in respect of buildings, motive power, plant and machinery, including in the latter term everything in the nature of an engine, gas making plant, tramway plant, electrical generating machines and all the subsidiary works required. The necessity to exercise careful control over such outlay arises from the fact that as the element of a probable appreciation in value decreases, it is requisite to provide for the very opposite conditions, namely, a probable fall in value due to two causes, first, a gradual wasting of the asset due to wear and tear (which cannot be met by current repairs and renewals charged to revenue account) and the further probability that future advances in scientific and mechanical knowledge may result in the discovery of new and improved methods long before the original plant, etc., is worn out and the loan repaid. Local authorities as well as commercial concerns are here confronted with a difficult problem and have carefully to consider whether it is advisable to discard the

in the case of original outlay only. The case is different when once the original loan has been repaid and the asset becomes the property of the local authority free from any debt and without the necessity to set aside any annual instalment, or to pay any interest upon the loan. It should here be remembered that under the sinking fund method of repayment both these annual charges are a burden upon the revenue or rate account of each year of the repayment period, and that this annual burden is equal during the whole of the period, seeing that although part of the loan may have been repaid out of the sinking fund, yet the interest upon the amount of loan so repaid must be charged to the revenue or rate account and added to the fund. On the final repayment of any loan the revenue or rate account is immediately relieved of a heavy annual charge, consisting of the instalment and interest upon the loan, and the local authority is in possession of an undertaking which has been provided out of the revenue or rate of previous years, and in addition maintained in a state of efficiency by annual repairs and renewals, and very possibly kept up-to-date by improvements defrayed by means of additional charges to revenue or rate.

It is therefore equitable to assume that it is obligatory upon future generations of ratepayers to ensure that this asset shall be maintained by them in an efficient state, as far as possible, but since any expenditure upon repairs and renewals cannot prevent a further loss in value, such wastage should be made good by charging future years with an annual sum in respect of depreciation. This is a matter which is frequently overlooked, but it is worthy of serious consideration. When, in spite of all repairs and renewals, the asset becomes valueless, or so nearly so that it cannot be worked at a profit or economically, it must be replaced and the depreciation fund in hand may then be applied in relief of the cost of the new works, leaving only the balance to be raised by further borrowing.

Section VII.

The Equation of the Period of Repayment

CHAPTER XXXII.

THE EQUATION OF THE PERIOD OF REPAYMENT OF LOANS REPAYABLE AT VARIOUS DATES WHICH ARE REQUIRED TO BE REDEEMED ON ONE UNIFORM DATE:—

1. WHERE THE LOANS ARE AUTHORISED IN RESPECT OF OUTLAYS OF VARYING CHARACTER, EACH HAVING A DIFFERENT LIFE OR PERIOD OF CONTINUING UTILITY AND CONSEQUENT REPAYMENT.
2. WHERE THE NECESSITY TO FIND THE EQUATED PERIOD OF REPAYMENT ARISES ON THE CONSOLIDATION OF EXISTING LOANS.

THE ARITHMETICAL METHOD OF FINDING THE EQUATED PERIOD KNOWN AS THE EQUATION OF PAYMENTS, THE TRUE OR MATHEMATICAL METHOD; AND THE ERROR IN THE GENERALLY ADOPTED ARITHMETICAL METHOD.

THE NECESSITY FOR THE EQUATION OF THE PERIOD OF REPAYMENT. In the early days of municipal loans they were relatively small in amount as compared with what they are at the present day, and as a general rule each loan was sanctioned for a specific purpose and related to one class of outlay only. When a sanction or authorisation included several classes of outlay a definite amount of loan was authorised, and a definite period was prescribed, for each class, carrying out the provision in Section 234 (1) of the Public Health Act of 1875, namely:—

> "Money shall not be borrowed except for permanent works (including under this expression any works of which the cost ought, in the opinion of the Local Government Board, to be spread over a term of years)."

Under this Act (Sec. 234 [4]) the period of repayment may be fixed by the local authority with the sanction of the Local Government Board.

Section 243 of the same Act dealing with loans to local authorities by the Public Works Loan Commissioners, provides:

"That in determining the time when a loan under this section shall be repayable the Local Government Board shall have regard to the probable duration and continuing utility of the works in respect of which the same is required."

With the widening of the sphere of municipal activity to include gas works, tramways, electric supply, hydraulic power supply and other manufacturing (and in many cases profit earning) utilities, the problem became more complicated, seeing that the total loan authorised for any one undertaking necessarily included outlays of very diverse character having widely varying periods of utility and consequently varying periods of repayment. Further difficulties were introduced, when, under the Public Health Acts Amendment Act, 1890, local authorities generally were empowered, subject to certain conditions laid down in the Stock Regulations of 1891, etc., to raise money by the issue of stock redeemable on a specified date or at the end of a given number of years.

During the earlier years, when each loan was authorised for one class of outlay only with a definite repayment period, all that was necessary was to keep a separate sinking fund for each loan, when the whole amount was borrowed in one year, and to keep a separate fund for each year's borrowings, when the loan was borrowed over a series of years. The same applied to loans authorised for one undertaking including various classes of outlay, each having a different period of repayment, so long as the sinking funds could be kept distinct for each class of outlay or each year's borrowings, and the funds could mature at the end of the respective periods and the loans then be redeemed. But when it became possible to raise loans by the issue of stock redeemable on a fixed date it at the same time became necessary to so arrange the sinking fund instalments that the total loan should be redeemed on the prescribed date irrespective of the repayment periods imposed for the several component parts of the outlay.

The difficulty is overcome by ascertaining the equivalent average date of repayment of the whole of the loan, and calculating the annual instalment required to be set aside and accumulated in one instead of in several sinking funds. The actual practice varies. In some cases the sanction states the specific amounts to be borrowed for each class of outlay with the period of repayment allowed for each class, and the duty of fixing the average date falls upon the local authority. In

other cases the local authority submits a scheme to the Local Government Board showing the various sums proposed to be borrowed for each class of outlay, the respective periods of repayment suggested and the proposed average date of repayment of the whole loan. This is subject to revision by the Government department concerned, especially as to the period desired by the local authority, and this being fixed the average or equated period is found by calculation in a manner which will be discussed in detail.

These are the general considerations which are involved in the equation of the period of repayment, but they may be further complicated by reason that the amounts are borrowed over a series of years, or that the loans in respect of the component parts of the outlay are borrowed together at irregular times, and without any definite allocation as between the various classes of outlay. In many cases it is necessary to set aside temporary instalments during construction, leaving the final instalment to be ascertained by adjustment when the total loan has been borrowed and the whole of the works carried out and an apportionment made of the outlay. In the case of very large undertakings this cannot be done until the engineer has given his final certificate.

Another difficulty arises in cases where the operation of the sinking fund is suspended for a number of years, and it is often almost impossible, owing to a combination of the above factors, to decide upon the amount of the first instalment to be set aside. The only permanent factor is the repayment of the whole of the loan on a fixed date. The same considerations apply on the consolidation of several existing loans repayable at various dates, when it becomes necessary to fix a uniform date of repayment and adjust the instalment, having regard to the amounts now in the several sinking funds.

Space will not permit of the detailed treatment of any such examples owing to the difficulty of stating a set of conditions which would be generally applicable. Each case must be dealt with on the individual facts, but any question likely to arise may be treated by one or more of the methods described in this book. As a general rule, where the conditions are at all complicated, it is better to set aside, during the construction period, temporary instalments of a general nature and defer any final adjustment until the whole of the loan has been borrowed and the actual outlay under each head has been certified by the engineer.

THE METHODS OF FINDING THE EQUATED PERIOD OF REPAYMENT. The foregoing remarks will now be illustrated by the following example which is of a simple character without any of the complications previously referred to, and relates to a loan of £56,000, raised by the issue of stock redeemable at par in one sum on a date to be ascertained. The loan is required for an undertaking comprising outlay of a variable nature, each class of which has a different life or period of utility, and separate periods are prescribed for each. In order to ascertain the date of redemption of the stock it is required to find the equated period corresponding to the several prescribed periods and amounts. This method will apply to ordinary loans if it be required to repay the total debt on one date.

The classes of outlay, the amounts of loan authorised in respect of each class and the prescribed periods of repayment are as follows: the rate of accumulation is 3 per cent. per annum. The rate of interest payable upon the stock does not enter into the calculation.

TABLE XXXII. A.

PARTICULARS OF THE LOAN OF £56,000.

Nature of outlay.	Amount of loan authorised.	Prescribed period.
Class A	£10,000	repayable in 45 years.
,, B	20,000	,, ,, 29 ,,
,, C	24,000	,, ,, 15 ,,
,, D	2,000	,, ,, 5 ,,
	£56,000	

THE ARITHMETICAL METHOD. Under ordinary circumstances the above amounts of loan would be repayable at the end of the respective periods by means of the usual sinking fund instalments, as described in previous chapters, but under the present conditions the whole of the loan is repayable on one date, which has to be so fixed that the lender will receive his money at a time equivalent to that at which he would have received it if the original varying periods and amounts had been adhered to. In arithmetic this is known as "the equation of payments," and the rule is stated as follows:—

Multiply each debt by the number of years which will elapse before it becomes payable; add the results together; divide this sum by the sum of the debts; the quotient will be the number of years in the equated time.

But it is stated in the books on arithmetic that this is only approximately correct, and can only be taken as equitable when the various times of repayment are not widely apart. The error, it is pointed out, is in favour of the payer as it extends the period of repayment. This arithmetical method will first be applied to ascertain the equated period of repayment of the above loan of £56,000, after which an investigation will be made in order to ascertain the true equated period suggested in the arithmetic book. This is the more necessary because in the case of the loans of local authorities the various times of repayment are very widely apart. The result of the investigation into the true equated period will show that it is shorter than under the arithmetical method. In the case of a local authority, however, the arithmetically equated period may be preferred because it is slightly in favour of the payer (in this case the revenue or rate account of the equated period) as it extends the repayment beyond the time required by the true equated method. The effect of equating several sinking fund periods is to reduce the total period over which the repayment is spread and thereby relieve part of the original period of any charge whatever. The burden of this relief is thrown upon the equated period taken as a whole, and any extension of this period tends to redress the inequality caused by the equation. With regard to the interest upon the loan, which will be considered fully in Chapter XXXV, it should be remembered that under the original conditions the annual interest charge to revenue or rate will gradually be reduced as the loans with shorter prescribed periods are repaid; whereas under the generally adopted method of distributing the annual burden after equation, interest upon the full amount of the loan is payable equally during and charged equally against the revenue or rate account of each year of the equated period. This will be discussed in a later chapter, but the present subject of enquiry relates solely to the method of finding the true equated period.

The calculation of the equated period relating to the above loan of £56,000 will now be made, by the arithmetical rule, and, although the figures adopted give a period of an exact number of years, yet in practice this will rarely be obtained. It is in fact somewhat difficult to state original conditions which will work out to an even number of years in the equated period.

TABLE XXXII. B.

THE ARITHMETICAL METHOD OF FINDING THE EQUATED PERIOD OF REPAYMENT.

Required the equated period, at the end of which the total loan should be repayable, corresponding to the repayment of the component parts of the loan at the end of the respective periods prescribed for each.

Nature of outlay.	Amount of loan authorised.	Prescribed periods.	Product of amount of loan multiplied by number of years.
Class A	£10,000	45 years	£450,000
,, B	20,000	29 ,,	580,000
,, C	24,000	15 ,,	360,000
,, D	2,000	5 ,,	10,000
	£56,000		£1,400,000

Equated period:—

$$\frac{1,400,000}{56,000} = 25 \text{ years.}$$

THE TRUE OR MATHEMATICAL METHOD. The correctness of the above arithmetically equated period will now be investigated, as well as the effect of the alteration upon the repayment of the loans.

To do this it is first necessary to ascertain the exact equivalent of the original conditions. This will be stated as if those conditions had been carried out by setting aside an equal annual instalment in respect of each of the amounts of loan and accumulating them in four separate sinking funds to repay the several portions of the loan at the end of 5, 15, 29, and 45 years respectively. But as the individual sinking funds mature at different dates each annual instalment must be reduced to its present value. The sum of such present values represents the amount of money now required to purchase an equivalent annuity or annual instalment for the equated period of 25 years. For purposes of the comparison to be made in Chapters XXXIV and XXXV it is necessary to know the individual instalments to be set aside during the whole of the above periods, and the annual instalment as well as its present value will therefore be shown in each case. The calculations are all similar to others which have been previously worked out so that it is not necessary to show the actual working as in earlier chapters.

TABLE XXXII. C.

THE EQUATION OF THE PERIOD OF REPAYMENT OF LOANS, repayable at various dates, which are required to be redeemed on one uniform date.

Loan of £56,000, authorised for outlays of varying character each having a different life or period of continuing utility, and a consequent period of repayment.

Rate of accumulation, 3 per cent.

Annual instalments required under the original conditions.

Equated period for a loan for public works consisting of outlay having varying lives or periods of continuing utility.

Nature of outlay	Period allowed for repayment	Amount of loan authorised. Details	Amount of loan authorised. Total.	Annual instalment to repay loan.	Present value of Loan repayable at end of period.	Present value of Future annual instalments.
Class A	45 years	—	10,000	107·85	2644·39	2644·39
,, B	29 ,,	—	20,000	442·29	8486·93	8486·93
,, C	15 ,,	—	24,000	1290·40	15404·69	15404·69
,, D	5 ,,	—	2,000	376·71	1725·22	1725·22
Total	45 years	—	56,000	2217·25	28261·23	28261·23

The present values in the above table are the present values both of the amounts of loan repayable at the end of the respective periods and also of the corresponding sinking fund instalments, since the instalments, if accumulated, will, at the end of the respective periods, amount to the respective loans.

The whole of the loans, although repayable at the end of successive periods, have now been reduced to a common measure, namely, a "present value" of £28261·23, which represents the amount for which the various sinking fund obligations might be redeemed at the present time, and upon which the calculation of the true equated period will be based.

The following argument is summarised in Table XXXII. D., which may be referred to with advantage:—

If the arithmetical calculation of the equated period of 25 years be correct this sum of £28261·23 should in 25 years, at 3 per cent., amount to £56,000, and the annuity which it will purchase (or the sinking fund instalment) should also amount to £56,000 in that period. This, however, is not the case.

It may be found by calculation that £28261·23 will in 25 years, at 3 per cent., amount to £59172·75, or an excess of £3172·75. And it may also be ascertained that the annuity or annual instalment, which will amount to £59172·75 in 25 years, at 3 per cent., and of which £28261·23 is the present value, is £1622·98 per annum. Since £28261·23 is also the present value of the four annual instalments required to provide the component parts of the loan of £56,000 at the end of the respective periods of 5, 15, 29 and 45 years, it is obvious that the error lies in the number of years in the equated period, as found by the arithmetical method.

The next step is to calculate the actual annual instalment (and also the present value of the instalment) required to repay £56,000 in 25 years, the equated period as found by the arithmetical method, but which there is reason to suspect is in excess of the true period. It will be found, on making the calculation, that the annual instalment to repay £56,000 in 25 years is £1535·95, and its present value £26745·90 (which is also the present value of £56,000 due at the end of 25 years). This annual instalment of £1535·95 cannot, of course, be compared with the four sinking fund instalments, amounting together to £2217·25, to be accumulated for the original periods because they are all for different numbers of years, but it has been ascertained that they are equivalent to an annual instalment of £1622·98 to be set aside for 25 years, and accumulated at 3 per cent. It is therefore possible to compare the two annual instalments of £1535·95 and £1622·98, and the result is to prove that the arithmetically equated period gives an annual instalment which is less by £87·03 than the exact equivalent of the original instalments. In other words, the arithmetically equated period is too long. The following table (XXXII. D.) shows the above conclusions:—

TABLE XXXII. D.

THE TRUE OR MATHEMATICAL METHOD OF FINDING THE EQUATED PERIOD OF REPAYMENT.

Showing the annual instalments and their present values under

(1) the original conditions;

(2) the arithmetically equated period;

(3) the true or mathematically equated period.

Sinking fund instalment and present value thereof for	Amount of loan.	Number of years.	Sinking fund instalment per annum.	Present value of loan or sinking fund instalment.
	10,000	45	107·85	2644·39
Original periods of	20,000	29	442·29	8486·93
repayment	24,000	15	1290·40	15404·69
	2,000	5	376·71	1725·22
	56,000		2217·25	28261·23
25 years, the equated period as found by the arithmetical method:—				
Amount of loan ...	56000·00	25	1535·95	26745·90
Amount which will be provided at end of 25th year	59172·75	25	1622·98	28261·23
Surplus	3172·75	25	87·03	1515·33
24 years:—				
Amount of loan ...	56,000	24	1626·65	27548·28
23 years:—				
Amount of loan ...	56,000	23	1725·58	28374·73
Amount which will be provided at end of 23rd year	55,776	23	1718·68	28261·23
Deficiency ...	224	23	6·90	113·50

Having ascertained the exact error in the annual instalment under the arithmetical method of equation the error in the equated period itself may now be found. It has already been ascertained that at 3 per cent. £26745·90 will amount to £56,000 in 25 years, and the problem is to ascertain in how many years £28261·23 will amount to £56,000. As the present value of the four original instalments, namely, £28261·23, is greater than £26745·90, which is the present value of £56,000, it will amount to £56,000 in a smaller number of years than 25. The exact number of years may be ascertained by using the formula relating to Table I, in standard calculation form, No. 1, to find the amount of £1 in any number of years, but this will give a result consisting of a number and a fraction. In cases such

as the present the exact fraction of the year is required only for the purpose of fixing the nearest number of whole years, so that the problem will work out in practice. The method of finding the number of years by using the formula relating to Table I is as follows, and may be compared with the standard calculation form for the purpose given in Chapter X.

STATEMENT XXXII. E.

Required the number of years in which £28261·23 will amount to £56,000 at 3 per cent.

By formula and logs.

$A = PR^N$ $\qquad$ $56{,}000 = 28261{\cdot}23 \times 1{\cdot}03^N$.

Log. amount at end of period	...	56,000	4·7481880
deduct Log. present value	...	28261·23	4·4511911
$= \text{Log. } R^N$	...		0·2969969
divide by Log. R = 1·03	...		0·0128372

To divide one Log. by another find the Logs. of the above Logs. as if they were actual numbers, viz.:

Log. 2969969 = 6·4727516
Log. 128372 = 5·1083703

1·3642813

which is the Log. of the number of years, viz. ... 23·136

In order to avoid the necessity of dividing one log by another, the exact number of years may be ascertained by means of Thoman's Table giving the logs of R^N, at 3 per cent., for various years, as follows:—

Proceed as in the above Statement by deducting the log of the present value of the annual instalments under the original conditions, from the log of the amount of loan repayable at the end of the period. The remainder is the log of R^N, from which the value of N, may be obtained. In the above case, the log of R^N is ... ... 0·2969969

On referring to Thoman's Table, the nearest logs of R^N, above and below this, are found to be as follows:—

at 3 per cent., log R^N	24 years	0·3080934
	23 years	0·2952562
	a difference of	0·0128372

The next step is to find the difference between the log. of R^N, as found in the above calculation in Statement XXXII. E., and from which it is required to find the value of N; and the lower of the above logs in Thoman's Table as follows:—

log of R^N, in calculation, as above	0·2969969
log of R^N, 23 years, by Thoman	0·2952562
a difference of	0·0017407

and the fraction of a year above 23 years is:—

$$\frac{17407}{128372} \text{ or } 0{\cdot}135598 \text{ as may be found by logs.}$$

The number of years therefore is 23·136, and agrees with the calculation in Table XXXII. E., made by means of the formula.

Another method of making the calculation, after having found the above difference of 0·0128372 in the logs, is to refer to the tables of differences given in the margin of the ordinary log tables, and under 128 the following amounts will be found:

$$\begin{aligned} \cdot 10 &= 0{\cdot}12800 \\ \cdot 07 &= 0{\cdot}00900 \\ \cdot 004 &= 0{\cdot}00051 \\ \hline \cdot 174 &= 0{\cdot}13751 \text{ of 1 year,} \end{aligned}$$

which differs from the previous result by less than 1 day.

Summary of the True or Mathematical Method. In order to ascertain the number of years in the true equated period it is advisable to find, first, the approximate number of years by the arithmetical method, in this case 25 years, and then to find by calculation in the manner already described, and shown in Table XXXII. C., the present value of the several annual instalments under the original conditions before equation; in the present instance, £28261·23. The next step is

to find the amount of loan, £59172·75, which will be provided by the accumulation, at the estimated rate, of the above present value, for a number of years (25) equal to the equated period, as ascertained by the arithmetical method. The amount of loan which will be thereby provided should then be compared with the actual amount of the loan. As a general rule the amount of loan which will be provided by the accumulation of the present value of the annual instalments under the original conditions before equation at the end of the equated period, will be greater than the amount of the loan, and will denote that the equated period as found by the arithmetical method is in excess of the true equated period.

The enquiry is therefore confined to the present value of the actual loan, at the estimated rate of accumulation, for periods of years less than the number of years (25) as found by the arithmetical method.

Reference is next made to the tables of compound interest in order to ascertain the present value of the loan at the estimated rate of accumulation for periods less than the arithmetically equated period.

Reverting to the present example, a period of 24 years will first be taken, and it will be found by calculation on standard calculation form, No. 2, that the present value of £56,000 due at the end of 24 years is £27548·28, requiring an annual instalment (as may be found by standard calculation form, No. 3x) of £1626·65.

The above present value, £27548·28, as compared with £28261·23, the actual present value of the original annual instalments, before equation, is still insufficient, and a period of 23 years is adopted. Similar calculations will show that the present value of £56,000 due at the end of 23 years, is £28374·73, which is very nearly correct. And therefore 23 years is adopted as the nearest to the true equated period. The future annual instalment to be spread equally over the equated period may now be ascertained by calculation on standard form 3x, and will be found to be £1725·58.

The only conclusion which may properly be drawn from the above facts, is, that an annual instalment of £1725·58 to be accumulated for 23 years at 3 per cent., is, within a small limit of error, the true mathematical equivalent of the four annual instalments under the original conditions, amounting together to £2217·25, as shown in Table XXXII. C., to be accumulated for the respective periods shown in that table. It has nothing whatever to do with the incidence of the burden upon the

revenue or rate accounts of the equated period, which will be fully considered in a later chapter.

The correct figures as to the equated period of repayment of the above loan are therefore as follows:—

Amount of loan repayable at the end of 23 years	£56,000
Present value thereof	£28374·73
Annual instalment of which £28374·73 is the present value, and which will amount to £56,000 in 23 years at 3 per cent.	£1725·58

Owing to the fact that the equated period is fixed at the nearest whole number of 23 years, instead of 23·136 years, as shown in Statement XXXII. E., the annual instalment of £1725·58 is larger than the instalment £1718·68, which is the equivalent of the present value £28261·23, of the original instalments shown in Table XXXII. C. The following table shows the error involved by taking the nearest whole number of years:—

23 years 3 per cent. based upon .—	Capital sum.	Present value.	Annual instalment.
Actual amount of loan	£56,000	£28374·73	£1725·58
Actual present value of the original instalments	£55,776	£28261·23	£1718·68
Difference	£224	£113·50	£6·90

The above table shows that £28261·23 will not amount to £56,000 in 23 years but only to £55,776, requiring an annual instalment of £1718·68, consequently it is not possible to arrive at anything nearer than an approximation of the period. The annual instalment to be set aside for 23 years corresponding to the present value of £28261·23, namely £1718·68, is less by £6·90 than the instalment required to repay £56,000, and would fall short of repaying the loan by £224 at the end of 23 years. The method is an approximation only and in actual practice the arithmetical method would give a number of years containing a fraction, but the result is sufficiently correct if the nearest number of even years be taken.

The effect of adopting an equated period of 25 years, as shown by the arithmetical method, instead of 23 years as shown

by the true equated method, may be seen from an inspection of Table XXXII. D. It may be taken as a general rule that the arithmetical method gives the longer repayment period, and relieves the revenue or rate accounts of the equated period as compared with the true method of equation which should always be used when it is desired to accelerate the repayment of the loan, or when extreme accuracy is required.

FURTHER PROOF. The previous example of the equation of the period of repayment of a loan of £56,000 is not a case occurring in actual practice. The amounts of outlay composing the loan as well as the periods of repayment are all assumed, and the problem has been treated purely from the theoretical standpoint in order to show the difference between the arithmetical and true methods of finding the equated period. The basis of the method there adopted is to ascertain, first, the annual instalments required in respect of each part of the loan and then to find the present value of such instalments. The same present values may be obtained in one operation by finding the present values of the several amounts of loan repayable at the ends of the respective periods, where it is not necessary to know the actual instalments, as was the case in the foregoing example.

A further example will now be given, using figures occurring in actual practice, but with a shorter period of ultimate repayment than 45 years, and the results will be compared with the previous example. The calculation will be made by the shortest possible method. The actual example may be found in the report by a Select Committee of the House of Commons upon the Repayment of Loans by Local Authorities (1902), page 261. This example of an equated period was put in evidence by the Assistant Secretary of the Local Government Board to illustrate the method adopted by the Board in order to arrive at the average period to be granted for the repayment of a loan to be expended upon a gas undertaking where the component parts of the outlay have a variable probable duration and continuing utility.

The following table shows, in the first four columns, the nature of the outlay, the period allowed for repayment in respect of each, and the amount of loan to be expended in each case. The fourth column shows the component parts of the loan in respect of which the same period is allowed. The details are taken from the appendix to the above report. In order to avoid repetition the table also contains the present

values of the component parts of the loan as found by calculation. The annual instalments are not shown, because in this case they do not enter into the calculation. This table may be compared with Table XXXII. C.

TABLE XXXII. F.

THE EQUATION OF THE PERIOD OF REPAYMENT OF LOANS repayable at various dates, which are required to be redeemed on one uniform date.

Loan of £9105, authorised for outlays of varying character, each having a different life, or period of continuing utility, and a consequent period of repayment.

Rate of accumulation, 3 per cent.

Equated period for a loan for Gas-works purposes.

Nature of outlay.	Period allowed for repayment.	Amount of loan authorised.		Annual instalment to repay loan	Present value of	
		Details.	Total.		Loan repayable at end of period.	Future annual instalments.
Buildings	30 years	2500·00		In this case the method of finding the true equated period is based upon the present value of the loan instead of upon the present value of the annual instalment		
Mains	,,	1245·00				
Gasometer	,,	1500·00				
Condensers	,,	530·00				
	30 years	———	5775·00		2379·22	2379·22
Purifiers	20 years		1000·00		553·67	553·67
Benches	15 ,,		1200·00		770·23	770·23
Meters	10 ,,		530·00		394·37	394·37
Retorts	2 ,,		600·00		565·55	565·55
	30 years		9105·00		4663·04	4663·04

The report shows also the arithmetical method adopted to arrive at the equated period of repayment of the total loan authorised, and this method corresponds exactly with the method laid down in the books on arithmetic and illustrated by Table XXXII. B. The actual working is given in the report and may be summarised as follows:—

TABLE XXXII. G.

THE ARITHMETICAL METHOD OF FINDING THE EQUATED PERIOD OF REPAYMENT OF THE LOAN.

Nature of outlay.	Amount of loan authorised.	Prescribed periods.	Product of amount of loan multiplied by number of years.
Buildings, etc. ...	5,775	30 years	173,250
Purifiers	1,000	20 ,,	20,000
Benches	1,200	15 ,,	18,000
Meters	530	10 ,,	5,300
Retorts	600	2 ,,	1,200
	9,105		217,750

Equated period:—

$$\frac{217750}{9105} = 23{\cdot}915, \text{ or 24 years.}$$

THE TRUE OR MATHEMATICAL METHOD OF FINDING THE EQUATED PERIOD OF REPAYMENT. As already stated, the method about to be described differs slightly from that adopted in the previous example, and is shorter. The first step is to find by calculation on standard calculation form, No. 2, the present values of the component parts of the loan for the respective periods allowed. These present values are shown in the sixth column in Table XXXII. F., and amount together to £4663·04. The next step is to find the number of years in which £4463·04 will amount to the loan of £9,105, at 3 per cent., in order to replace a gradual repayment of the component parts of the loan by a simultaneous repayment of the whole. In the previous example, three methods are described of finding the number of years, one being by direct calculation by means of the formula relating to the amount of £1 per annum in standard calculation form, No. 3, which is illustrated by Statement XXXII. E. The second method of finding the number of years is by means of Thoman's tables, and is fully described in the previous example. The third method is by trial and error, based upon the approximate number of years in the equated period found by the arithmetical method, and this method will be applied to the present instance. Using standard calculation form No. 1, it may be found that £4663·04, accumulated at 3 per cent., will amount to:—

in 24 years	£9479·00
in 23 years	£9202·90
in 22 years	£8934·87

as compared with the equated period as found by the arithmetical method:—

in 24 years	£9105·00

It is obvious that the true period is nearer to 23 years than to 22 years. If it be calculated exactly by either of the methods used in the previous example it will be found to be 22·638 years; and therefore 23 years should be adopted in practice instead of 24 years as found by the arithmetical method above described. The actual difference between the periods found by the two methods is 1·277 years. In the previous example the actual difference was 1·864 years, but in that case the longer repayment period was assumed to be 45 years, whereas in the present instance it is 30 years only. There is not any common ratio existing between the original and the equated periods, or between the two equated periods as found by the arithmetical and true methods. The number of years in the equated period depends upon the interaction of the component parts of the loan and the respective periods prescribed for their repayment.

Having found the true equated period in the above manner the enquiry strictly comes to an end, but if the annual instalment is required to be spread equally over the period it may be found in the usual manner on standard calculation form No. 3x. So far as the lender is concerned this is quite equitable, but having regard to the varying life of the assets created out of the loan the question of the annual charges to revenue or rate during the period becomes important and will be fully considered in the following chapters.

Section VIII.

The Equation of the Incidence of Taxation

CHAPTER XXXIII.

THE EQUATION OF THE INCIDENCE OF TAXATION.

COMPARISON OF THE TOTAL ANNUAL LOAN CHARGES TO REVENUE OR RATE, BEFORE AND AFTER THE EQUATION OF THE PERIOD OF REPAYMENT, SHOWING THE UNEQUAL INCIDENCE OF TAXATION IF THE ANNUAL INSTALMENT AND INTEREST UPON THE TOTAL LOAN BE SPREAD EQUALLY OVER THE EQUATED PERIOD.

The subject of enquiry in the previous chapter is the correct method of finding the equated date of repayment of several loans repayable at varying dates, and the result of such enquiry is to show that the generally adopted arithmetical method is wrong in principle seeing that it tends to prolong the period. Having found the equated date the next step is to ascertain the annual instalments to be charged to revenue or rate account during the equated period in substitution for the annual instalments required under the original conditions before equation. The present practice is to regard the matter purely from the point of view of the loan holder and to set aside an equal annual instalment to be spread over the whole of the equated period without any regard to the incidence of taxation or the life of the asset created out of the loan. Seeing, however, that the annual instalments are accumulated in the sinking fund and are not repaid to the lender until the end of the period it is immaterial to him how the annual instalments are distributed over the revenue or rate accounts of the equated period. On the contrary it is a matter of concern to the ratepayer that the annual contributions out of revenue or rate are borne equitably by successive years, and this question will now be considered. The permanent character of the security for local loans is shown by the preferential nature of the redemption of part of the loan out of the sinking fund before maturity. In the case of financial and commercial undertakings any such redemptions are made pro rata or by some method in which each loanholder has an equal chance.

The effect of the generally adopted method of equation of the period of repayment is to reduce the annual instalment

during the earlier years of the equated repayment period and thus relieve the revenue or rate account of those years. This may at first sight appear strange, when it is borne in mind that under an equation of the period the total loan is repaid at an earlier date although the mathematical result may be exactly equal.

This will be found to be the case on referring to Table XXXII. D. in Chapter XXXII, giving the annual instalments required to repay a loan of £56,000 in an equated period of 23 years in substitution for periods of 5, 15, 29 and 45 years.

The following table (XXXIII. A.) shows the annual instalments to be set aside, dividing the original periods into five, at the end of four of which, part of the loan would have been repaid, whereas the total loan is repayable at the end of the 23rd year under the equated method:—

TABLE XXXIII. A.

Loan of £56,000 (authorised for outlays of varying nature having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Comparison of the annual charges to revenue or rate in respect of the annual instalments under (1) the original conditions, and (2) after equation where such annual instalments are spread equally over the equated period.

	(1) Original periods.		(2) Equated period.		Equated period as compared with original periods.	
Periods of equal incidence.	No. of years.	Sinking fund instalments.	No. of years.	Sinking fund instalments.	Increase.	Decrease.
5 years	5	2217·25	5	1725·58	—	491·67
10 years	10	1840·54	10	1725·58	—	114·96
8 years	8	550·14	8	1725·58	1175·44	—
6 years	6	550·14	—	—	—	550·14
16 years	16	107·85	—	—	—	107·85
	45		23			

The above table shows that during the first five years of the equated period the annual instalment is reduced by £491·67, and that during the second period of 10 years there is a similar

annual reduction of £114·96. The heaviest charge falls upon the third and final period of eight years, which is part of an original period of 14 years. During this period the annual instalment is greater by £1175·44 than the corresponding annual instalment under the original conditions before equation. This large increase in the annual instalment is due to the fact that under the original conditions, before equation, £26,000 of loan would have been repaid by the end of the 15th year, being the end of the second portion both of the equated and original periods. This amount of loan, having a short period of repayment, naturally required a larger annual instalment than the remaining loans having longer periods of repayment. After the final repayment of the loan, at the end of the equated period of 23 years, by means of the equal annual equated instalment of £1725·58, the revenue or rate account is relieved of all contributions both in respect of the annual instalment and interest upon the loan.

As already pointed out, the actual figures in individual cases will vary in accordance with the amounts of the respective loans and the length of the various periods allowed for repayment, but the generality of equations will follow the main features here outlined. The results of an equation of the period of repayment may be summarised as follows:—As regards the annual sinking fund instalment to be charged to revenue or rate account, the earlier and later years of the original repayment period will be relieved and the resulting burden thrown upon the middle portion of the original repayment period, which is the final part of the amended equated period. This relief will of course apply in full to that part of the original repayment period beyond the equated period, seeing that the whole of the loan will then have been repaid.

The following table, XXXIII. B., shows the result of the equation of the period of repayment, as regards the interest upon the loan, chargeable against the revenue or rate account of each year of the equated period as compared with the corresponding annual interest charges under the original conditions, before equation. Under the original conditions the loan would have been gradually repaid, thereby reducing the annual interest charges against the revenue or rate accounts of subsequent years, but after the equation of the period of repayment, the revenue or rate account of each year of the equated period is charged with interest upon the total amount of the loan. The equation of the annual charge for interest upon the loan will be fully considered in Chapter XXXV.

TABLE XXXIII. B.

Loan of £56,000 (authorised for outlays of varying nature having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Comparison of the annual charges to revenue or rate in respect of interest upon the loan under (1) the original conditions, and (2) after equation.

Periods of equal incidence.	(1) Original periods.		(2) Equated period.		Equated period as compared with original periods.	
	Loan.	Interest.	Loan.	Interest.	Increase.	Decrease.
5 years	56000	1960	56000	1960	—	—
10 years	54000	1890	56000	1960	70	—
8 years	30000	1050	56000	1960	910	—
6 years	30000	1050	—	—	—	1050
16 years	10000	350	—	—	—	350
45 years						

The foregoing tables show that under the generally adopted equated method the relief during the first period of 5 years is solely in respect of the annual sinking fund instalment and that the annual interest charges are unaltered owing to the fact that no part of the loan is repayable, under the original conditions, until the end of the fifth year. During the second period of 10 years there is a decrease in the annual instalment, but there is an increase in the amount of the annual interest charges, because the repayment of £2000 of loan which, under the original conditions, would have been made at the end of the fifth year, has by the equation of the period of repayment been deferred until the end of the 23rd year. The third period of 8 years, being the final portion of the equated period, has to bear an increased annual charge of £2085·44, being an increase in the annual instalment of £1175·44 in addition to an increased annual interest charge of £910.

The explanation of the large increase in the total annual burden imposed upon this period is, that it has to bear, not only the relief to the earlier periods of 5 and 10 years, but also the total relief to that part of the original repayment period which is beyond the equated period. The whole of the foregoing conclusions are shown in the following table:—

TABLE XXXIII. C.

Loan of £56,000 (authorised for outlays of varying nature, having prescribed periods of repayment) the whole to be redeemed in one sum at the end of an equated period.

Showing the variation in the total annual charges to revenue or rate in respect of the sinking fund instalment and interest upon the loan, under (1) the original conditions and (2) after equation, where such annual instalments are spread equally over such equated periods.

A Summary of Tables A and B above.

Periods of equal incidence.	Sinking fund instalment.		Annual interest on loan.		Total charge to revenue or rate.	
	Increase.	Decrease.	Increase.	Decrease.	Increase.	Decrease.
5 years	—	491·67	—	—	—	491·67
10 years	—	114·96	70·00	—	—	44·96
8 years	1175·44	—	910·00	—	2085·44	—
6 years	—	550·14	—	1050·00	—	1600·14
16 years	—	107·85	—	350·00	—	457·85
45 years						

These results are so remarkable that some enquiry may profitably be made into the matter, not only as to the necessity to make the equation, but also as to the effect of the equation upon the incidence of taxation. The necessity to make an equation of the period may arise in several ways. The most important is in order to provide for the repayment of a loan raised by the issue of stock redeemable on a fixed date where the loan is authorised for works having varying lives or periods of utility. In the case of a loan to provide for outlay of one character only, or for different classes of outlay having the same life or duration of utility, there is not any necessity to make an equation, the calculation being a simple one. The governing factor is the unequal life or duration of utility of the works authorised, upon which are based the periods allowed for the repayment of the component parts of the loan.

A further need for the equation of the period of repayment arises on the consolidation of several loans repayable at various dates. This may be part of a large financial scheme undertaken

with the object of generally simplifying the finances of a local authority or on the issue of stock to replace a number of small loans borrowed for short periods. The issue of such a stock avoids the necessity of reborrowing a large number of small sums continually falling due, and gives a permanency to the outstanding debt. It also considerably simplifies the sinking fund book-keeping and renders much easier not only the investment of the sinking fund but also the redemption of part of the loan during the operation of the fund. Further, investors prefer a stock of large amount which is quoted in the Stock Exchange list and is readily saleable. Looking at the matter from the investor's point of view, it is a coincidence perhaps that the equated period generally found necessary is about 20—30 years, which is as long as investors generally approve. Both the shorter and longer repayment periods allowed for the repayment of local debt are not suitable for permanent investment as a stock, and local authorities are thus obliged to rely upon the small investor who causes much more administrative work than the holders of a stock. The renewal or reborrowing of small loans falls upon the officials of the local authority, whereas the burden of any change in the ownership of a registered stock is borne by the holder except the registration of the transfer and the preparation of the new certificate.

The investor may therefore be eliminated from the enquiry because if he is willing to accept payment on the equated date the arithmetical or mathematical methods of ascertaining that date both give a sufficiently near approximation. The investor, except in a very academic way, is not concerned with the annual charges for redemption of the loan. The effect of the equation of the period upon the incidence of taxation therefore becomes the principal subject of enquiry, and the above table (XXXIII. C.) shows that there is a very wide difference as between the original and the equated periods in regard to the burden imposed upon successive years or periods of years. It is here advisable to recapitulate the principles governing the method of fixing the original periods of repayment. The predominant factor in fixing the proper repayment periods to be allowed in respect of each individual class of outlay is found in the principle laid down in the Statutes and adopted in the practice of Parliament and the Government departments, namely, that all loans shall be repaid during the period of utility or duration of the works in respect of which the loan was borrowed. But a local authority has not any capital and can only repay the loan by annual contributions out of rate or out

of the profits of its revenue earning undertakings. It may be contended that revenue earning undertakings should be treated in a different manner to purely spending departments, such as a sanitary, highway or education authority, where the annual expenditure, both for current expenses and debt redemption charges, is taken direct from the pockets of the community by way of a rate. Here it is important to adjust the incidence of taxation very accurately—and this is the object of the careful scrutiny by Parliament and the Government departments of the periods of repayment allowed. The effect of this scrutiny is seen in the original repayment periods allowed which are generally fixed at a number of years well within the life of the works for which the loan is authorised. If these periods are properly allowed and the sinking fund instalments are based upon them there is an equitable incidence of the annual burden, and the annual instalments may then properly be considered as the equivalent of an annual charge for depreciation—thereby carrying out the principle laid down in an earlier chapter of making each ratepayer contribute annually his due proportion of the cost of the benefits he receives each year, whether that cost be paid for during the year or be spread over a series of years.

But the equation of the period of repayment is purely a financial operation, and relates solely to the date of repayment of the loan without any regard to the effect of such equation upon the annual charges to the community by way of rate. The case is different with a commercial or financial undertaking where the repayments of debt are made out of the general assets of the concern and are not charged against the profit and loss account except and in so far as the operations of each individual year cause a loss of capital due to wear and tear of the asset. The repayment of debt and the annual charge to revenue are in the case of such undertakings kept severely separate and distinct. In the case of a local authority the conditions are the exact opposite. In the first place, there is a careful and searching enquiry by Parliament and the Government departments, with the object of fixing the annual amounts to be charged to the revenue or rate accounts of successive years in respect of the repayment of the debt and the consequent charges for interest. These total charges are in many cases regularly met out of revenue or rate during a part of the period so allowed, and it may then become necessary or advisable to make an alteration in the date at which the loan shall be repaid, and an equation of the period of repayment is made

resulting in such a drastic rearrangement of the total annual charges that the original careful calculations as to the life of the asset are ignored and rendered valueless. Reverting to the present example, it will be seen from Table XXXIII. C. that, although, in consequence of the equation of the period, the final repayment of the loan is expedited, there is actually a decrease in the annual burden for the next 15 years and an absolute relief from any burden whatever during the later years of the original period which were, under the conditions existing at the time the loan was authorised, charged with their due proportion. And the whole of the added burden is imposed upon the final years of the newly ascertained or equated period at a time when probably the undertaking may have to incur outlay on renewals.

If it at any time becomes necessary or advisable to expedite the repayment of the loans the calculation of the equated period, and the resulting amended annual instalment should be made in such a manner as to impose a proportionate part of the additional burden upon each year of the equated period of repayment, instead of, as is the present practice, relieving both the earlier and the later years of the original repayment period at the expense of the middle portion of that period which is also the final portion of the equated period. A method of doing this as regards the annual instalment will be described in Chapter XXXIV, and as regards the interest upon the loan in Chapter XXXV. Up to this point the general question of the repayment of debt has been treated from an actuarial or mathematical standpoint only, but the disturbing element now introduced by the necessity to accelerate the final repayment of the loan and to vary the dates of repayment, by substituting therefor a common date for the repayment of the whole of the loan, depends upon circumstances which are generally of a variable and somewhat accidental nature. It is therefore necessary to find an equitable practical method of restoring the original status, namely, to charge the revenue or rate account of each year with its due proportion of the annual burden of redemption and interest charges. At this stage it becomes advisable to differentiate between the annual charges in respect of the redemption of debt and the interest payable upon the loan, because where loans are repaid by means of a sinking fund, interest is payable upon the total amount of the loan during the whole of the redemption period of whatever duration, and ceases entirely at the end of that period. Any reduction therefore in the period of redemption will correspondingly

reduce the period during which there is any charge whatever in respect of interest upon the loan, and it will, in the first instance, be assumed that it is perfectly equitable to ignore the relief of later years in respect of interest upon the loan due to an increased sinking fund burden imposed upon the earlier years in consequence of the accelerated final extinction of the debt. Confining the enquiry therefore solely to the sinking fund instalment, annually charged to the revenue or rate account, it is necessary to revert to the primary factor in the redemption of debt, namely, the life or duration of continuing utility of the asset created out of the loan. A broad line is here required to be drawn between the two objects of the annual instalment, namely, the repayment of the debt and the charge to revenue or rate of the cost of the asset during its life or period of utility.

Under the original conditions before equation the total debt would have been gradually repaid at the end of a series of periods up to the 45th year, whereas under the amended conditions the total loan will be repaid in one sum at the end of the 23rd year or approximately in about one half the number of years. The magnitude of the annual instalment to be set aside and charged to revenue or rate in order to redeem a given loan depends primarily upon the period allowed for its redemption, which is based upon the life of the asset; and therefore if the total period be reduced, the periods allowed for the repayment of the component parts of the loan should be correspondingly reduced.

CHAPTER XXXIV.

THE EQUATION OF THE INCIDENCE OF TAXATION (*Continued.*)

THE ANNUAL INSTALMENT.

THE VARIOUS METHODS OF ADJUSTING THE ANNUAL CHARGES TO REVENUE OR RATE DURING THE EQUATED PERIOD IN PROPORTION TO THE LIFE OR DURATION OF CONTINUING UTILITY OF THE ASSET CREATED OUT OF THE LOAN, VIZ:—BY CHARGING THE REVENUE OR RATE ACCOUNT OF EACH YEAR OF THE EQUATED PERIOD WITH THE ANNUAL INSTALMENT CHARGEABLE AGAINST EACH YEAR, BEFORE EQUATION. AND IN ADDITION THERETO A SUPPLEMENTARY ANNUAL INSTALMENT:—

(*a*) TO BE SPREAD EQUALLY OVER THE EQUATED PERIOD, OR

(*b*) TO BE PROPORTIONATE, YEAR BY YEAR, TO THE ANNUAL INSTALMENTS BEFORE EQUATION.

The previous argument will now be applied to the example under review, namely, the repayment of a loan of £56,000, authorised for large public works, the component parts of which have varying periods of continuing utility and consequent prescribed periods of repayment. Under the original conditions the repayment of the loan was spread over a period of 45 years, but under the altered conditions it is required that the whole of the loan shall be repaid at the end of an equated period of 23 years. In this chapter the correct method of spreading the actual burden equitably over the equated period is the subject of enquiry and not the method of finding the true equated period which has been fully discussed in Chapter XXXII.

The object is to distribute the annual sinking fund burden equitably over the reduced period of repayment instead of imposing an undue burden upon the final years of the equated period which is the effect of the method generally adopted, as shown in Table XXXIII. C.

For this purpose it will be an advantage to tabulate the original conditions in the example considered in Chapter

XXXII, and to show also the proportionate amended periods of repayment under the amended conditions due to the equation of, or alteration in, the final period of repayment, as follows:—

TABLE XXXIV. A.

Loan of £56,000 (authorised for outlays of varying nature, having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Comparison of the annual charges to revenue or rate in respect of the annual instalments under (1) the original conditions, and (2) after equation if such instalments are increased in proportion to the reduction in the repayment period.

Rate of accumulation, 3 per cent.

		Number of years allowed for repayment of loan.		Annual sinking fund instalments. 3 per cent.	
Class of outlay.	Amount of loan authorised.	Original conditions	Equated period of 23 years	45 year period.	23 years equated period
A	10,000	45	23	107·85	308·14
B	20,000	29	14	442·29	1170·53
C	24,000	15	8	1290·40	2698·95
D	2,000	5	3	376·71	647·06
	56,000			2217·25	4824·68

The above annual instalments under the 23 year equated period will repay the component parts of the loan at the end of the respective reduced periods of repayment, and are not in any way equated instalments. They are placed here only to show the effect of reducing the period from 45 to 23 years. They illustrate forcibly the wide difference between the financial obligation to repay the loan and the anticipated life of the various classes of outlay. Both these factors are distinct, but the generally adopted method of fixing the future annual instalment after ascertaining the equated period does not make any such distinction but treats them as being equivalent.

It may be at once stated that it is not possible to adjust the unequal burden of the annual instalment as ascertained by the generally adopted method, by reducing each of the periods allowed for the component parts of the outlay in proportion to the reduction in the final period of repayment as shown in the above table. There are so many disturbing factors and the conditions are so widely altered by reducing the periods by

one half that the results obtained show wider differences than exist under the method now in use. The author has worked out the problem in detail, but the results are too long to give in full and would not be of any practical value.

It is therefore necessary to adopt another line of enquiry in order to find a method of determining the annual instalment or instalments to be charged to revenue or rate to repay the total loan in one sum at the end of the equated period. So far as the investor is concerned the equated period of repayment already found is comparatively correct, but a more important matter is the equated annual incidence of the burden upon each year's revenue or rate account. So long as the period of repayment and the life of the asset are the same the two methods yield identical results, but any alteration in the term of repayment makes an important difference in the annual charges to revenue or rate in successive years. By the method now generally adopted and previously described the annual instalment is spread over the whole of the equated period of repayment, and this is considered to be an equitable substitute for a gradually decreasing charge which has been ascertained after careful enquiry as to the life of the asset.

Very little consideration will show that this is very far from being correct; and the result of the previous enquiry is to show that although the effect of the equation is to reduce the period of repayment and correspondingly increase the total charge for redemption, yet there is actually a reduction in such annual charge during the earlier years of the equated period and an absolute relief from any charge whatever during that part of the original period beyond the equated period. These reductions in the charges against the earlier and later years of the original period involve a severe additional annual burden upon the later years of the equated period. Although this inequality is not fully appreciated yet its effect has been mentioned in several places in the reports of the parliamentary committees which have enquired into the finances of local authorities where it is pointed out that under an equated method, loans for outlays for which short terms are generally allowed are not repaid until the end of the longer equated period, and consequently further borrowing powers ought not to be granted when the asset is exhausted.

The result of the previous discussion of the subject is to emphasise the fact that as regards the annual charge to revenue or rate the most important factor is the life of the asset, and it may naturally be concluded that if any change is made in the

final period of repayment the amended annual instalment to be charged to revenue or rate account should continue to bear as near as possible an approximate ratio to the original charge, instead of, as is the present practice, spreading the burden equally over the equated period without any regard to the life of the asset.

A method of doing this will now be fully described, taking as an example the loan of £56,000 already used to illustrate the previous remarks in Chapter XXXII upon the method of finding the true equated period. In comparing the two methods it is important to bear in mind that where the periods are not equated the several sinking funds will mature at successive dates, at each of which portions of the original loan will be repaid out of such funds, whereas under the equated method the whole loan is repayable on one date. Further, the equated period covers the date of final repayment of one or more of the component parts of the original loan. In the present chapter the sinking fund instalment only will be considered, leaving out of account for the moment the interest payable upon the loan which, as already pointed out, ceases entirely at an earlier date under the equated method, although the later years of the equated period bear a larger interest charge than they do under the original conditions. This is shown by Table XXXIII. B.

The following method of adjusting the annual charge to revenue or rate after an equation of the period is based upon the relative periods allowed for the repayment of the component parts of the loan, as expressed by the sinking fund instalments originally found requisite to repay the several portions of the loan at the end of the respective periods prescribed. In order to make the adjustment it is first requisite to ascertain the amount which would have been in the fund if the original instalments had been allowed to accumulate until the end of the equated period of 23 years instead of repaying £2,000 at the end of the 5th year and a further £24,000 at the end of the 15th year. The original annual instalments are as follows:—

for the first five years	£2217·25	per annum.
for the next ten years	1840·54	,,
for the final eight years... ...	550·14	,,

and the following table shows the amount which would be in the fund at the end of the 23rd year under the above conditions.

STATEMENT XXXIV. B.

Loan of £56,000 (as above).

Showing the amount which will be in the sinking fund at the end of the equated period of 23 years if the original annual instalments as shown in Table XXXIII. A., are set aside for the periods originally prescribed and no part of the fund is applied in repaying the loan.

Interest at 3 per cent.

1. Amount of £2217·25 per annum for 5 years	11772	
Amount thereof at the end of a further 18 years		20040
2. Amount of £1840·54 per annum for 10 years	21100	
Amount thereof at the end of a further 8 years		26728
3. Amount of £550·14 per annum for 8 years		4892
Total amount in the fund at end of 23rd year		51660

being:—

Loan repayable at end of 5th year ...	2000	
accretions for 18 years	1404	
		3404
Loan repayable at end of 15th year ...	24000	
accretions for 8 years	6403	
		30403
Annual instalment of £550·14 accumulated for 23 years		17853
Amount as above		51660

The above amount of £51,660 which would have been in the fund if the original annual instalments had been added thereto and allowed to accumulate for 23 years, represents that portion of the loan of £56,000 which would have been repaid by means of the annual charges to the revenue or rate accounts of the 23 years. These annual instalments are based upon the respective repayment periods proper to be allowed for the component parts of the outlay and may be accepted as fair and proper charges against each year's revenue or rate, irrespective of the equated period allowed for the repayment of the loan.

TABLE XXXIV. C.

Loan of £56,000 (authorised for outlays of varying nature, having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Comparison of the annual charges to revenue or rate in respect of the annual instalments under (1) the equated method generally adopted, and (2) in which the annual instalments as originally ascertained are supplemented by an equal additional instalment spread over the equated period.

Periods of equal incidence.	Original annual instalments.	Additional annual instalment.	Total annual instalments.	Annual instalment as equated.
5 years	2217·25	133·73	2350·98	1725·58
10 years	1840·54	133·73	1974·27	1725·58
8 years	550·14	133·73	683·87	1725·58
—				
23 years				
—				

By the aid of the above figures a comparison will now be made between the annual instalments obtained by the three methods, namely:—

(1) Instalments payable during the original prescribed periods, providing for the gradual repayment of the loan at the end of 5, 15, 29, and 45 years, and which are based upon the life of the asset.

(2) Instalments payable during the equated period only, based upon an equal annual charge to the revenue or rate account of each year, according to the method generally adopted. In this case the life of the asset is not taken into account in fixing the annual burden. It is true that the periods originally prescribed enter into the arithmetical calculation of the equated period, but the effect of this is lost by spreading the instalment equally over the whole of the equated period so ascertained.

(3) Instalments payable during the equated period only, but which are not equal throughout the period but are based upon the life of the asset and are approximately proportionate to the instalments before equation.

In the following table the instalments (2) and (3) are compared with (1) those found requisite under the original conditions, and the increase or decrease in the annual charge is shown in respect of each period of equal incidence. It will be noticed that the final 8 years of the equated period alone bear any increased charge.

TABLE XXXIV. D.

Loan of £56,000 (authorised for outlays of varying nature, having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Comparison of the annual charges to revenue or rate in respect of the annual instalments:—

1. Based upon the original repayment periods.
2. The equated method generally adopted.
3. The annual instalments, as in Table XXXIV. C.

The following increased or decreased annual charges are compared with the amounts in Column 1.

	Original periods.	Equated method usually adopted.		Equated method previously described.	
Periods of equal incidence.	Annual instalment.	Annual instalment.	Increase+ Decrease−	Annual instalment.	Increase+ Decrease−
5 years	2217·25	1725·58	−491·67	2350·98	+133·73
10 years	1840·54	1725·58	−114·96	1974·27	+133·73
8 years	550·14	1725·58	+1175·44	683·87	+133·73
6 years	550·14	—	−550·14	—	−550·14
16 years	107·85	—	−107·85	—	−107·85
45 years					

The above table shows that the method just described, although it concerns the annual instalment only, removes the gross inequality which exists in the method generally adopted, in which each year of the third period of 8 years is charged with £1175·44 per annum more than is the case under the original conditions before equation, based upon the life of the asset. In the method above described, not only is there a decreasing annual charge due to the fact that the classes of outlay with shorter periods of utility are written off in the earlier years but the total relief to the final 22 years of the original period (or the post equated period) is charged against the whole of the equated period. This is almost as near an equalisation of the original annual burden as can be made, and removes the objection that under the generally adopted method of equation the redemption of loans authorised for outlay in respect of which only short periods are granted is unduly delayed. With regard to the annual charges in respect of interest upon the loan, there is not any variation from the conditions previously shown under the generally adopted

method of equation, and Table XXXIII. B., showing the comparison will still apply.

At this point it is interesting to compare the total annual charges for sinking fund instalment and interest upon the loan by means of the following table which may usefully be compared with Table XXXIII. C.:—

TABLE XXXIV. E.

Loan of £56,000 (authorised for outlays of varying nature having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Showing the variations in the total annual charges to revenue or rate under the method described in Table XXXIV. C., as compared with the original annual instalments before equation.

This table should be compared with Table XXXIII. C.

Periods of equal incidence.	Sinking fund instalment. Increase.	Sinking fund instalment. Decrease.	Interest on loan. Increase.	Interest on loan. Decrease.	Total charge to revenue or rate. Net Increase.	Total charge to revenue or rate. Net Decrease.
5 years	133·73	—	—	—	133·73	—
10 years	133·73	—	70	—	203·73	—
8 years	133·73	—	910	—	1043·73	—
6 years	—	550·14	—	1050	—	1600·14
16 years	—	107·85	—	350	—	457·85
45 years						

The principal points to be noticed in the above table as compared with Table XXXIII. C., which shows the difference between the total annual loan charges under the original conditions and under the generally adopted equated method, are (1) the reduction in the excess additional charge during the third period of 8 years from £2085·44 per annum to £1043·73 per annum due solely to the reduction in the sinking fund instalment, and (2) the additional burden imposed upon the first 15 years. As already stated it has been assumed that it is perfectly equitable to consider interest upon the total loan as a proper charge against revenue or rate during the later years, although the increase in the annual interest is due to the delay in the repayment of the loan for purely financial reasons. The following statement shows the final repayment of the loan by means of the instalments in Table XXXIV. C., ascertained in the above manner.

STATEMENT XXXIV. F.

Loan of £56,000 (as above).

Showing the final repayment of the loan by the operation of the sinking fund at the end of the equated period of 23 years, by setting aside the original annual instalments based upon the life of the asset, and a further additional instalment spread equally over the equated period. Such instalments are shown in Table XXXIV. C.

1. Amount of £2350·98 per annum for 5 years	12482	
Amount thereof at the end of a further 18 years	———	21249
2. Amount of £1974·27 per annum for 10 years	22633	
Amount thereof at the end of a further 8 years	———	28670
3. Amount of £683·87 per annum for 8 years		6081
Amount of loan		56000

The amended annual instalments to repay £4340, the balance of loan unprovided for by the accumulation of the original instalments before equation, should properly be distributed over the equated period in such a manner that they will be proportionate to the original instalments, but they may be spread equally over the equated period without any great injustice being caused. Table XXXIV C. shows that the additional annual instalment under these conditions is £133·73. It is, however, necessary to point out the correct method, in order that it may be applied to cases where the magnitude of the loan renders it desirable to make an absolute equation of the incidence of the sinking fund instalment, as well as of the number of years over which the equated burden should be spread. If the original periodically decreasing annual instalments are set aside for the respective repayment periods, and are allowed to accumulate until the end of the equated period they will provide £51,660 of the original loan of £56,000, leaving £4,340 to be provided by the accumulation of supplementary annual instalments to be set aside for similar numbers of years and allowed to accumulate for the same periods. The

first step in the calculation is to divide the £4,340 in the same proportions as the £51,660 as follows:—

(1)	5 years.........	£20,040	£1683·6
(2)	10 years.........	26,728	2245·4
(3)	8 years.........	4,892	411·0.
	23 years.........	£51,660	£4340

The component parts of the £4,340 represent the amounts which will be in the fund at the end of the period due to (1) an annual instalment to be set aside for 5 years and then accumulated for a further 18 years, followed by (2) an annual instalment to be set aside for the next 10 years and then accumulated for a further 8 years, followed by (3) an annual instalment to be set aside for the final 8 years at the end of which period the fund will mature. This latter period of 8 years is the one which bears the undue burden under the method of equation generally adopted and which it is the object of the present adjustment to remove. Having analysed the component parts of the deficiency of £4,340, the respective instalments are ascertained by working backwards. In the case of item (1) it is required to ascertain the amount of an annuity for 5 years, which amount if accumulated for a further 18 years will provide £1683·6. The first step therefore is to find by standard calculation form No. 2 the present value of £1683·6 due at the end of 18 years at 3 per cent., and having done so to find by standard calculation form No. 3x the annuity for 5 years which will amount to this sum. The calculation may be made direct by Thoman's method as follows:—

CALCULATION XXXIV. G.

To find the annual instalment to be set aside and accumulated for a given number of years, at the end of which period the amount thereof will continue to accumulate for a further specified period and will then amount to a given sum.

Required, the annuity for 5 years which will amount to [the present value of £1683·6 due at the end of a further 18 years].

By Thoman's Tables and Logs.

First period 5 years. Second period 18 years.

Log of the given future sum … …	1683·60	3·2262330
deduct Log. R^N, 3 per cent. 18 years		0·2310700
Log of … … … … …	988·92	2·9951630
add Log. a^n, 3 per cent. 5 years …		9·3391623
		12·3343253
deduct Log. R^N, 3 per cent. 5 years + 10		10·0641861
Log of annuity required … … …		2·2701392
Annual instalment required …		£186·26

Note. This calculation may be compared with XVI. D. 1 and XXVII. C.

The second item may be ascertained in a similar manner, but the third calculation consists merely of finding the annual instalment, and it may be performed on standard form No. 3x. It is not necessary to give the actual details of the calculations, but merely to state the results in the following table which shows the manner in which the above deficiency of £4,340 will be provided at the end of the equated period.

STATEMENT XXXIV. H.

Loan of £56,000 (as above).

Showing the supplementary annual instalments to be set aside and added to the sinking fund during the equated period of 23 years, to repay £4,340 of the original loan unprovided by the original annual instalments added to the fund.

The following supplementary annual instalments are proportionate to the original annual instalments based upon the life of the asset, and are not equal during the whole of the repayment period, as was the case in Table XXXIV. C., and Statement XXXIV. F.

1. Amount of £186·26 per annum for 5 years	988·92	
Amount thereof at the end of a further 18 years. Calculation XXXIV. G.	———	1683·60
2. Amount of £154·62 per annum for 10 years	1772·60	
Amount thereof at the end of a further 8 years	———	2245·40
3. Amount of £46·22 per annum for 8 years		411·00
		4340·00

The above annual instalments may be usefully compared with those previously obtained where the supplementary annual instalment of £133·73 is spread equally over the equated period, as shown in Table XXXIV. C. The following table shows the annual instalments under the present method:—

TABLE XXXIV. J.

Loan of £56,000 (authorised for outlays of varying nature having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Showing the annual charges to revenue or rate in respect of the annual instalments under (1) the equated method generally adopted, and (2) in which the annual instalments as originally ascertained are supplemented by additional annual instalments spread over the equated period in proportion to the original periods allowed.

Periods of equal incidence.	Original annual instalment.	Additional annual instalment.	Total annual instalment.	Annual instalment as equated.
5 years	2217·25	186·26	2403·51	1725·58
10 years	1840·54	154·62	1995·16	1725·58
8 years	550·14	46·22	596·36	1725·58
23 years				

It is not necessary to give details of the actual calculations, but the following statement has been prepared in order to show the final repayment of the loan at the end of the equated period of 23 years, by means of the annual instalments in Table XXXIV. J., thereby proving the accuracy of the above method.

STATEMENT XXXIV. K.

Loan of £56,000 (as above).

Showing the final repayment of the loan by the operation of the sinking fund at the end of the equated period of 23 years by annual instalments spread over the equated period, with due regard to the life of the asset instead of being spread equally over such period.

Table XXXIV. J.

1. Amount of £2403·51 per annum for 5 years	12760·6	
Amount thereof at the end of a further 18 years	——	21724
2. Amount of £1995·16 per annum for 10 years	22872·3	
Amount thereof at the end of a further 8 years	——	28973
3. Amount of £596·36 per annum for 8 years		5303
Amount of loan		56000

Four methods have now been shown by which the loan of £56,000 may be repaid before and after the equation of the period, and the total annual loan charges for instalment and interest, under each method, will now be summarised.

TABLE XXXIV. L.

Loan of £56,000 (authorised for outlays of varying nature having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Comparison of the total annual charges to revenue or rate during the several periods of equal incidence forming part of the original extended period of repayment, and now constituting the equated period, under the following methods:—

1. The generally adopted method of equation in which the instalment is spread equally over the equated period.
2. The instalments originally calculated before equation based upon the life of the asset.
3. The method in which the original annual instalments, based upon the life of the asset, are set aside during the equated period, and any deficiency is made good by a supplementary annual instalment spread equally over the equated period. Table XXXIV. C.
4. The method in which the instalments during the equated period are exactly proportional to the life of the individual assets and to the original annual instalments based thereon. Table XXXIV. J.

Annual instalments.

Period of equal incidence.	1. The equated method.	Method (2) above Table XXXIII, A.	Method (3) above Table XXXIV. C.	Method (4) above Table XXXIV. J.
5 years	1725·58	2217·25	2350·98	2403·51
10 years	1725·58	1840·54	1974·27	1995·16
8 years	1725·58	550·14	683·87	596·36

Interest upon the loan. See Table XXXIII. B.

Periods of equal incidence	1. The equated method	Method (2) above.	Method (3) above.	Method (4) above.
5 years	1960	1960	1960	1960
10 years	1960	1890	1960	1960
8 years	1960	1050	1960	1960

Total annual charges for instalment and interest on loan.

Periods of equal incidence.		1. The equated method.	Method (2) above.	Method (3) above.	Method (4) above.
5 years	Instalment	1725·58	2217·25	2350·98	2403·51
	Interest	1960·00	1960·00	1960·00	1960·00
	Total	3685·58	4177·25	4310·98	4363·51
10 years	Instalment	1725·58	1840·54	1974·27	1995·16
	Interest	1960·00	1890·00	1960·00	1960·00
	Total	3685·58	3730·54	3934·27	3955·16
8 years	Instalment	1725·58	550·14	683·87	596·36
	Interest	1960·00	1050·00	1960·00	1960·00
	Total	3685·58	1600·14	2643·87	2556·36
6 years	Instalment	—	550 14	—	—
	Interest	—	1050·00	—	—
	Total	—	1600·14	—	—
16 years	Instalment	—	107·85	—	—
	Interest	—	350·00	—	—
	Total	—	457·85	—	—

The conclusions to be drawn from the above results are, that the generally adopted method of equating the burden upon successive generations of ratepayers is unjust to the later years of the equated period seeing that the acceleration of the final repayment of the loan ought to impose a larger burden upon each year of the equated period. It is also obvious that on the contrary the generally adopted method of equation relieves the earlier years instead of increasing the annual charge during such years. The dates of repayment, as originally fixed before equation, were based upon the life of the asset, and this should not be lost sight of in amending the annual instalment after the equation of the period, as it is in fact ignored, in the generally adopted method in which the whole of the outlay is treated as having an equal repayment period. The proper and consistent method of apportioning the burden between the several years of the equated period is to adhere as closely as

possible to the original annual instalments before equation because by this means alone can two important results be attained, namely, the annual redemption charge to the revenue or rate account of each year will be proportionate to the wastage of the asset and, which is equally important, the loan in respect of outlay of short duration, if not actually repaid, will be in the sinking fund slightly earlier than the date at which it would have been repaid under the original conditions. This will remove an objection at present existing as to re-borrowing for outlay of short duration where the repayment of the loan has been delayed by the equation of the period. A final conclusion seems to be that in making the adjustment in the annual instalment the true mathematical method last described should be followed although it may involve rather more intricate calculations. The equation of the incidence of the annual interest charges will be considered in the following chapter.

CHAPTER XXXV.

THE EQUATION OF THE INCIDENCE OF TAXATION (*continued*).

INTEREST UPON THE LOAN.

THE METHOD OF ADJUSTING THE ANNUAL CHARGES TO REVENUE OR RATE DURING THE EQUATED PERIOD IN PROPORTION TO THE LIFE OR DURATION OF CONTINUING UTILITY OF THE ASSET CREATED OUT OF THE LOAN.

BY CHARGING THE REVENUE OR RATE ACCOUNT OF EACH YEAR OF THE EQUATED PERIOD WITH THE ANNUAL AMOUNT OF INTEREST PAYABLE BEFORE EQUATION, AND IN ADDITION THERETO A SUPPLEMENTARY ANNUAL AMOUNT PROPORTIONATE YEAR BY YEAR TO THE ANNUAL INTEREST CHARGES BEFORE EQUATION.

A GENERAL SUMMARY OF THE RESULTS OBTAINED IN CHAPTERS XXXIII, XXXIV AND XXXV.

The previous enquiry into the method of adjusting the annual incidence of the loan burden, after equation, is confined solely to the annual instalment to be charged to revenue or rate and added to the sinking fund. The result of the enquiry is to prove that the final 8 years of the equated period are made to bear not only the whole of the relief to the 22 years of the original period beyond the equated period, but a certain amount in relief of the earlier years of the equated period. The annual instalments during the final 8 years of the equated period are as follows:—

Before equation, under the original conditions
Table XXXIII. A. £550·14

After equation:
(1) by the method generally adopted
Table XXXIII. A. £1725·58

(2) by the true method just described
Table XXXIV. J. £596·36

The annual charge for interest upon the loan during the final 8 years of the equated period has already been referred to. Under the original conditions, before equation, this annual charge was £1,050 only, owing to the fact that £26,000 of loan had been repaid by the end of the 15th year. Under the equated method the whole of the loan is not repayable until the end of the equated period, and consequently the interest upon the total loan becomes an annual charge against revenue or rate during the whole of the equated period. The effect is to impose upon the final 8 years of the equated period an interest burden of £910 per annum in addition to the annual charge of £1,050 under the original conditions. The first period of 5 years is charged annually with the same amount of interest upon the loan under both methods; and the second period of 10 years is charged with an additional £70 per annum only, being interest upon £2,000 of loan which would otherwise have been repaid at the end of the first period of 5 years. It is therefore apparent that the final 8 years of the equated period bears the greater portion of the interest of which the final 22 years of the original period is relieved, in addition to the whole of the sinking fund instalment.

The interest charges against the revenue or rate accounts of the final 8 years of the equated period are as follows:—

Before equation, under the original conditions	Table XXXIII. B.	£1050
After equation, under the method generally adopted, and also under the true method, above described, if limited to the annual instalment only	Table XXXIV. L.	£1960

The total annual loan charges during the final 8 years of the equated period are therefore as follows (Table XXXIV. L.):

Before equation, under the original conditions:		
Instalment	£550·14	
Interest	£1050·00	
		£1600·14

After equation:—

(1) by the method generally adopted:		
Instalment	£1725·58	
Interest	£1960·00	
		£3685·58
(2) by the true method just described, relating to the annual instalment only:		
Instalment	£596·36	
Interest	£1960·00	
		£2556·36

The method just described removes the injustice to the final years of the equated period so far as the annual instalment is concerned, but leaves untouched the question of the interest upon the loan.

The advisability of making a similar adjustment with regard to the interest charges is a matter upon which opinion may be divided, and in order to elucidate the question it is necessary to state a few general propositions. The practice now adopted in the case of all original loans is to spread the repayment of the principal over a series of years commensurate to the life or period of continuing utility of the asset created out of the loan, with certain limitations as to assets of long continuing or permanent utility. Of the three alternative methods allowed by statute, only one, the instalment method, involves an unequal annual charge in respect of interest upon the loan. In the case of the annuity and sinking fund methods the total annual charges for principal and interest are equal throughout the period of repayment or redemption. But in both these cases interest is payable during such redemption period only, and of course ceases on the final repayment of the loan. In other words, the total annual loan charges, both for instalment and interest are spread over a period depending upon the life of the asset, and the annual charge to revenue or rate is the same in each year of the period. This principle is carried out, and is considered equitable, in the case of individual loans relating to outlay of one character only, repayable on fixed dates, and in respect of which separate sinking funds are or may be kept.

But the conditions are different in the case of the equation of the period of repayment whether the equation is made on the

consolidation of existing loans having varying unexpired periods of repayment, or whether it is made in respect of one loan, relating to outlay of a varied character, each class having different periods of continuing utility and consequent periods of repayment. The effect of the equation of the period as regards the loan holder has already been fully discussed in Chapter XXXII, where it has been ascertained that the arithmetical method of finding the equated period generally adopted, although wrong in principle, may perhaps be considered sufficiently correct.

Prior to an equation due to either of the events already described, the amount to be charged to the rate or revenue account of each year has already been, or may be, ascertained, and the result of any true equation should be that the future substituted total annual burdens are in proportion to the original obligations; any variations therefrom being due only to the substituted period imposed, and since the original annual burdens were not equal during the original periods they certainly should not be equal during the substituted period.

The equation of the incidence of the sinking fund instalment has been fully discussed in Chapter XXXIV, and a method described in detail of finding the future instalments only. If it be required to adjust the annual incidence of the interest upon the loan with a similar object in view, the adjustment cannot be made in quite the same manner, seeing that in this case an equal annual amount of interest has to be paid to the loan holder, but the burden of providing the interest has to be spread unequally over the period. The method is in fact a combination of the sinking fund and annuity methods of repayment of debt, with varying instead of equal annual charges to revenue or rate.

In order to simplify the method of adjustment and to bring it as near as possible to the method of equating the annual incidence of the sinking fund instalment, already described in Chapter XXXIV, it is necessary to reduce the interest upon the loan under both sets of conditions to its corresponding capital value at the end of the equated period of 23 years. The capital value might also be expressed in terms of the present value, but it would not be so convenient because the loan is repayable at a future date. The annual interest of £1,960, payable upon the total loan during the 23 years of the equated period, will therefore be treated as if it were allowed to accumulate at compound interest at 3 per cent. until the end of that period. The sum to which £1,960 per annum will then amount may be ascer-

tained in the usual way by standard calculation form, No. 3, and will be found to be £63607·70.

For the present purpose this amount may be treated in exactly the same manner as the total loan of £56,000 when adjusting the annual incidence of the sinking fund instalment only. In the same way that it was there required to find three annual instalments of unequal amount, to be set aside and accumulated for successive periods of 5, 10, and 8 years respectively, it is now required to find three annual amounts of interest to be charged to the revenue or rate account, such annual interest charges to be of unequal amount during each of the above periods, and to bear a relation to the life of the asset as expressed in the repayment periods originally prescribed.

In the case of the adjustment of the annual instalment the basis of the calculation was the original annual instalments before equation and in the present example the basis is the original charges for interest upon the loan. In this connection it is important to point out, as will be seen on reference to the second column in the third part of Table XXXIV. L., that there is not, before equation, any definite ratio between the annual instalment and the annual interest charges during any period of equal incidence, consequently the two adjustments are quite distinct.

The next step is to ascertain the accumulated amount at the end of 23 years of the original annual interest charges before equation shown in Table XXXIII. B. in the same way that the amount in the sinking fund was ascertained at the same date by the accumulation of the original annual instalments shown in Statement XXXIV. B. as follows:—

TABLE XXXV. A.

Loan of £56,000 (authorised for outlays of varying nature having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Showing the accumulated amount at the end of the equated period of 23 years, of the original annual interest charges, as shown in Table XXXIII. B. (Interest at 3 per cent.)

This statement should be compared with XXXIV. B.

(1) Amount of £1,960 per annum for 5 years	10405·90	
Amount thereof at the end of a further 18 years	———	17715·40
(2) Amount of £1,890 per annum for 10 years	21666·70	
Amount thereof at the end of a further 8 years	———	27446·80
(3) Amount of £1,050 per annum for 8 years		9336·90
Total, being the accumulated amount of the original annual interest charges, before equation, at the end of the equated period		54499·10

Up to this point it has been ascertained that the accumulated amount, at the end of the equated period of 23 years, of the equal interest charges of £1,960 per annum after equation is	£63607·70
and the corresponding amount of the original varying annual interest charges as shown by the foregoing table	£54499·10
A deficiency of	£9108·60

which is exactly comparable with the deficiency of £4,340 in the case of the annual instalment (after Statement XXXIV. B.).

The adjustment of the present deficiency may be made by the method described in Chapter XXXIV leading up to Calculation XXXIV. G., and Statement XXXIV. H., but as the conditions as to period and rate per cent. are similar in both cases, and differ only in amount it is possible to adopt a shorter method by utilising the information there obtained and increase the supplementary annual charges found in Statement XXXIV. H., in the ratio that 4340 bears to 9108·60 as follows:—

TABLE XXXV. B.

Showing the method of finding the supplementary annual charges to revenue or rate to be added to the original annual interest charges before equation.

Periods of equal incidence.	Deficiency at end of 23 years. Annual instalment. Statement XXXIV. H.	Interest on loan. As above.
Amount of deficiency... ...	4340·00	9108·60

Supplementary annual charges to revenue or rate.		Annual instalments Table XXXIV. J.	Annual interest charges Table XXXV. C
5 year period	... ...	186·26	390·93
10 year period	... ...	154·62	324·51
8 year period	... ...	46·22	97·00
23 years	Total	387·10	812·44

The above annual interest charges are ascertained from the amounts in the first column by the ordinary rules of proportion or by logs., of which it is not necessary to show the actual working. The total annual interest charges to revenue or rate during the equated period may now be stated in the following table:—

TABLE XXXV. C.

Loan of £56,000 (authorised for outlays of varying nature having prescribed periods of repayment), the whole to be redeemed in one sum at the end of an equated period.

Showing the annual charges to revenue or rate in respect of interest upon the loan under (1) the equated method generally adopted, and (2) in which the annual interest charges originally payable are supplemented by additional annual amounts spread over the equated period in proportion to the original interest obligations.

This table should be compared with Table XXXIV. J.

Equated annual interest charges.

Periods of equal incidence.	Original annual interest charges.	Additional annual interest charges.	Total annual interest charges.	Annual interest charges under the equated method.
5 years	1960·00	390·93	2350·93	1960·00
10 years	1890·00	324·51	2214·51	1960·00
8 years	1050·00	97·00	1147·00	1960·00
23 years				

Statement XXXIV. K. shows the final repayment of the loan by means of the amended annual instalments to be spread over the equated period with due regard to the life of the asset, instead of being spread equally over such period, and thereby proves the accuracy of the method adopted with regard to the annual instalment. In a similar manner, although expressed in different terms, the following Statement XXXV. D. proves the accuracy of the method adopted in order to equate the incidence of the annual interest charges.

STATEMENT XXXV. D.

Loan of £56,000 (as above).

Showing that the accumulated amount of the amended annual interest charges ascertained as in Table XXXV. C. will be equal to the accumulated amount of the equal annual interest charges after equation, both at the end of 23 years, at 3 per cent. per annum.

This statement should be compared with Statement XXXIV. K.

(1) Amount of £2350·93 per annum for 5 years		
Amount thereof at the end of a further 18 years	———	21248·80
(2) Amount of £2214·51 per annum for 10 years		
Amount thereof at the end of a further 8 years	———	32159·40
(3) Amount of £1147 per annum for 8 years	———	10199·50
which is the accumulated amount of an annuity of £1960 for 23 years at 3 per cent. as previously ascertained		£63607·70

The above calculations (1) and (2) have been made direct by the "method by step" shown in Statement XXVII. C.

It will be gathered from the above Statement XXXV. D., that the annual interest charges to revenue or rate during the first two periods of 5 years and 10 years are greater than the annual amounts of interest payable to the loanholders during those periods after equation, as follows:—

5 years (2350·93—1960) an increase of 390·93

10 years (2214·51—1960) an increase of 254·51

and that the annual amounts of interest payable to the loanholders during the final 8 years of the equated period are greater than the amended annual amounts charged to revenue or rate during that period, in Statement XXXV. D., as follows:

8 years (1960—1147), a decrease of 813·00.

The correctness of the foregoing calculations is proved by the following statement giving the accumulated amounts of the above annuities at the end of the 23rd year, without details of the actual calculations which are similar to XXVII. C.:—

Amount of £390·93 per annum for 5 years, accumulated for a further period of 18 years at 3 per cent. per annum	£3533·40
Amount of £254·51 per annum for 10 years, accumulated for a further period of 8 years, at 3 per cent. per annum	£3696·00
	£7229·40
which is equal to the	
Amount of £813·00 per annum for 8 years at 3 per cent. per annum	£7229·40

This proves that the amounts charged to the revenue or rate account during the first 15 years, in excess of the amounts annually payable to the loanholders during that period, will, if accumulated, be sufficient to provide the future annual deficiencies in the amounts charged to revenue or rate account during the final 8 years of the equated period. It also points out the methods to be adopted as regards the actual book-keeping, and indicates the opening of an account which may be termed an:—

"Equated Loan Interest, Reserve Account,"

and which will closely resemble the repayment of a loan by an equal annual instalment of principal and interest combined, or the annuity method, but with a varying instead of an equal annual charge to revenue or rate. To the extent that the annual charges to revenue or rate are, during the earlier years, greater than the annual amounts payable to the loanholders by way of interest, the account will also partake of the nature of a sinking fund, and will therefore require the same careful future supervision as to the amount standing to the credit of the account, the rate of accumulation, and also the immediate preparation of a pro forma account showing the ultimate working out of the account.

As regards the actual book-keeping the above interest reserve account may be treated in two distinct ways, namely, by

crediting the account with the total annual amounts of interest charged to revenue or rate, as shown in Table XXXV. C. and debiting the account with the annual interest payable to the loanholders, which is the more scientific method as yielding an exact record of the actual transactions. The other method is to treat it as a reserve account pure and simple and credit it only with the above excess annual amounts of £390·93 and £254·21 charged to revenue or rate account during the periods of 5 and 10 years respectively. During the third period of 8 years the interest reserve account would of course be debited, and the revenue or rate account credited, with the difference of £813 per annum already referred to. Unlike a sinking fund proper, the amount to the credit of the interest reserve account need not be separately invested, but may be merged in the general assets of the undertaking, provided always that the proper annual amounts of interest at the calculated rate of accumulation are credited to the account, and charged to the current year's revenue or rate account. If the account be kept in this manner and compared annually with the pro forma account there should not arise at any time any necessity to make an adjustment so long as the repayment period remains unaltered. The following pro forma account will illustrate the method of keeping the interest reserve account applicable to the foregoing example:—

TABLE XXXV. E.

EQUATED LOAN INTEREST. RESERVE ACCOUNT.
Interest, 3 per cent. per annum.

Year.	Amount to credit at beginning of year.	Interest thereon.	Interest charged to revenue or rate.	Total credits.	Interest paid to loan holders.	Balance carried forward.	Year.
1	Nil	Nil	2350·93	2350·93	1960·00	390·93	1
2	390·93	11·73	2350·93	2753·59	1960·00	793·59	2
3	793·59	23·81	2350·93	3168·33	1960·00	1208·33	3
4	1208·33	36·25	2350·93	3595·51	1960·00	1635·51	4
5	1635·51	49·06	2350·93	4035·50	1960 00	2075·50	5
6	2075·50	62·26	2214·51	4352·27	1960·00	2392·27	6
7	2392·27	71·77	2214·51	4678·55	1960·00	2718·55	7
8	2718·55	81·56	2214·51	5014·62	1960·00	3054·62	8
9	3054·62	91·64	2214·51	5360·77	1960·00	3400·77	9
10	3400·77	102·02	2214·51	5717·30	1960·00	3757·30	10
11	3757·30	112·72	2214·51	6084·53	1960·00	4124·53	11
12	4124·53	123·73	2214·51	6462·77	1960·00	4502·77	12
13	4502·77	135·08	2214·51	6852·36	1960·00	4892·36	13
14	4892·36	146·77	2214·51	7253·64	1960·00	5293·64	14
15	5293·64	158·85	2214·51	7667·00	1960·00	5707·00	15
16	5707·00	171·21	1147·00	7025·21	1960·00	5065·21	16
17	5065·21	151·96	1147·00	6364·17	1960·00	4404·17	17
18	4404·17	132·12	1147·00	5683·29	1960·00	3723·29	18
19	3723·29	111·70	1147·00	4981·99	1960·00	3021·99	19
20	3021·99	90·66	1147·00	4259·65	1960·00	2299·65	20
21	2299·65	68·99	1147·00	3515·64	1960·00	1555·64	21
22	1555·64	46·67	1147·00	2749·31	1960·00	789·31	22
23	789·31	23·69	1147·00	1960·00	1960·00	Nil	23

A comparison has already been made, in Table XXXIV. L., between the total annual loan charges under the original conditions and under the generally adopted method after equation; and a final comparison will now be made between those methods and the one just described. The following results are worthy of careful study and show the very wide difference between the incidence of the total annual loan burden under the generally adopted method after equation on the one hand as compared with the annual charges under the original conditions before equation and also under the amended method just described. Under the method generally adopted the annual burden, both as regards the annual instalment and interest upon the loan, is spread equally over the whole of the equated period with a total disregard to the life of the asset and the consequent repayment periods originally based thereon. The following table shows that the original incidence of the burden is departed from under the generally adopted method after equation in that it relieves the earlier years, and throws a severe additional burden upon the later years, of the equated period. The result of the author's method is that the revenue or rate account of each year of the equated period is charged with an amount in respect both of annual instalment and interest upon the loan, which is exactly in proportion to the amount with which it would have been charged under the original conditions. These amended annual charges are greater than under the original conditions and include, as is equitable, the relief to the post equated period, and such relief is imposed rateably upon each year of the equated period instead of being charged against the later period of 8 years only.

TABLE XXXV. F.

The Equation of the Incidence of Taxation.

Showing the annual charges in respect of the sinking fund instalment, and interest upon the loan, during each year of the equated period.

A. Under the original conditions before equation.
B. Under the equated method as generally adopted.
D. Under the author's method of equation, relating to the annual instalment and interest upon the loan.

Note.—The above letters are used in the following charts.

	A TABLE XXXIV. L. Original conditions before equation.			B. TABLE XXXIV. L. Equated method generally adopted.			D. TABLE XXXIV. J. Equated method as described.		
eriods of equal incidence.	Annual instalment	Interest on loan.	Total.	Annual instalment	Interest on loan.	Total.	Annual instalment.	Interest on loan.	Total.
5 years	2217·25	1960·00	4177·25	1725·58	1960·00	3685·58	2403·51	2350·93	4754·44
0 years	1840·54	1890·00	3730·54	1725·58	1960·00	3685·58	1995·16	2214·51	4209·67
8 years	550·14	1050·00	1600·14	1725·58	1960·00	3685·58	596·36	1147·00	1743·36
Table	XXXIII. A.	XXXIII. B.		XXXIII. A.	XXXIII. B.		XXXIV. J.	XXXV. C.	

In order to make the foregoing results perfectly clear, three charts have been prepared as follows:—

THE EQUATION OF THE INCIDENCE OF TAXATION. CHART. I.

Showing the total annual loan charges in respect of the sinking fund instalment and interest upon the loan, during each year of the original and equated periods.

(A). Under the original conditions, before equation.

(B). Under the equated method, as generally adopted.

(C). Under the author's method of equation relating to the annual instalment only, as described in Chapter XXXIV.

(D). Under the author's method of equation relating to the annual instalment and interest upon the loan, as described in Chapter XXXV.

THE EQUATION OF THE INCIDENCE OF TAXATION. CHART. II.

Showing the total annual loan charges in respect of the sinking fund instalment and interest upon the loan, during each year of the original and equated periods.

(A). Under the original conditions, before equation.

(B). Under the equated method, as generally adopted.

(D). Under the author's method of equation relating to the annual instalment and interest upon the loan, as described in Chapter XXXV.

THE EQUATION OF THE INCIDENCE OF TAXATION. CHART III.

Showing the difference between the total annual loan charges in respect of the sinking fund instalment and interest upon the loan, during each year of the original and equated periods:—

(B). Under the equated method as generally adopted, in which the charge is spread equally over the period.

(D). Under the method described in Chapters XXXIV. and XXXV., in which the revenue or rate account is charged with annual sums based upon the life of the asset, and proportionate, year by year, to the annual charges before equation.

These charts show in graphic form:—

In Chart I, the total annual loan charges under each method, during each period of equal incidence. These annual charges are divided as between the interest upon the loan which is shown in the lower part of the diagram, and the annual instalment which is shown above it. The height of each column represents the total annual loan charges, and the width of the columns represents the number of years in the periods of equal incidence. This chart brings out clearly the comparatively small relief to the earlier years of the equated period and the large increased annual burden during the final eight years of such period.

In Chart II, the total annual charges under each method during each period of equal incidence are further compared, but without any subdivision as between the interest upon the loan and the annual instalment. The broken line shows the equal annual burden under the generally adopted method after equation. The thin unbroken line shows the annual burdens under the original conditions before equation, which were based, both as regards instalment and interest, upon the life of the asset. The thick unbroken line shows the corresponding annual charges under the author's method of equation. It will be noticed that the two unbroken lines agree very closely and differ widely from the broken line of the generally adopted method.

In Chart III, the total annual loan charges under the author's method of equation are taken as the standard or zero, and are compared, as to the equated period, with the charges under the generally adopted method after equation, and as to the post equated period with the charges under the original conditions. The area below the zero line represents, in the case of the equated period, the relief afforded by the generally adopted method after equation as compared with the author's method, and as regards the post equated period, the absolute relief afforded by the equation of the period irrespective of the method in which the burden is distributed over the equated period. The area above the zero line, which occurs only in the final 8 years of the equated period, represents the additional annual burden imposed upon this period under the generally adopted method as compared with the author's method. The actual amounts of relief and overcharge are taken from Table XXXV. F., and relate to the loan of £56,000 used to illustrate the problem, consequently any comparison based

solely upon that table must be made with this actual loan in mind. In order therefore to show the results in a form which will be readily appreciated, the above differences have each been expressed, in the chart, in terms of an annual rate. The basis upon which this has been done is a statement by a witness before one of the Parliamentary Committees appointed to enquire into such questions, who proved that in a particular case the immediate effect of an equation of the period was to reduce the rates by 3d. in the £ upon the annual value. This reduction was of course, only between the annual instalments, before and after equation, because during the earlier years of the equated period, as shown by Chart I, there is not any change in the amount of interest payable, no part of the loan having then been repaid by the maturing of the sinking fund for the shorter period. If, however, the comparison be made between the amount payable after equation and the proper amount which should have been payable under the author's method as a consequence of such equation, the saving would be 6·52 pence in the pound instead of 3 pence, and in the chart the relief to the first part of the equated period of 5 years has been taken at that figure. Upon this basis, the effect of an equation of the period, in the present instance, under the method generally adopted, is to relieve the annual rate accounts as follows:—

during the equated period:—

for a period of 5 years of 6·52 pence in the £.

for a period of 10 years of 3·20 pence in the £.

during the post equated period:—

for a period of 6 years of 9·76 pence in the £.

for a period of 16 years of 2·79 pence in the £.

and to impose an additional annual burden upon the final 8 years of the equated period of 11·85 pence in the £.

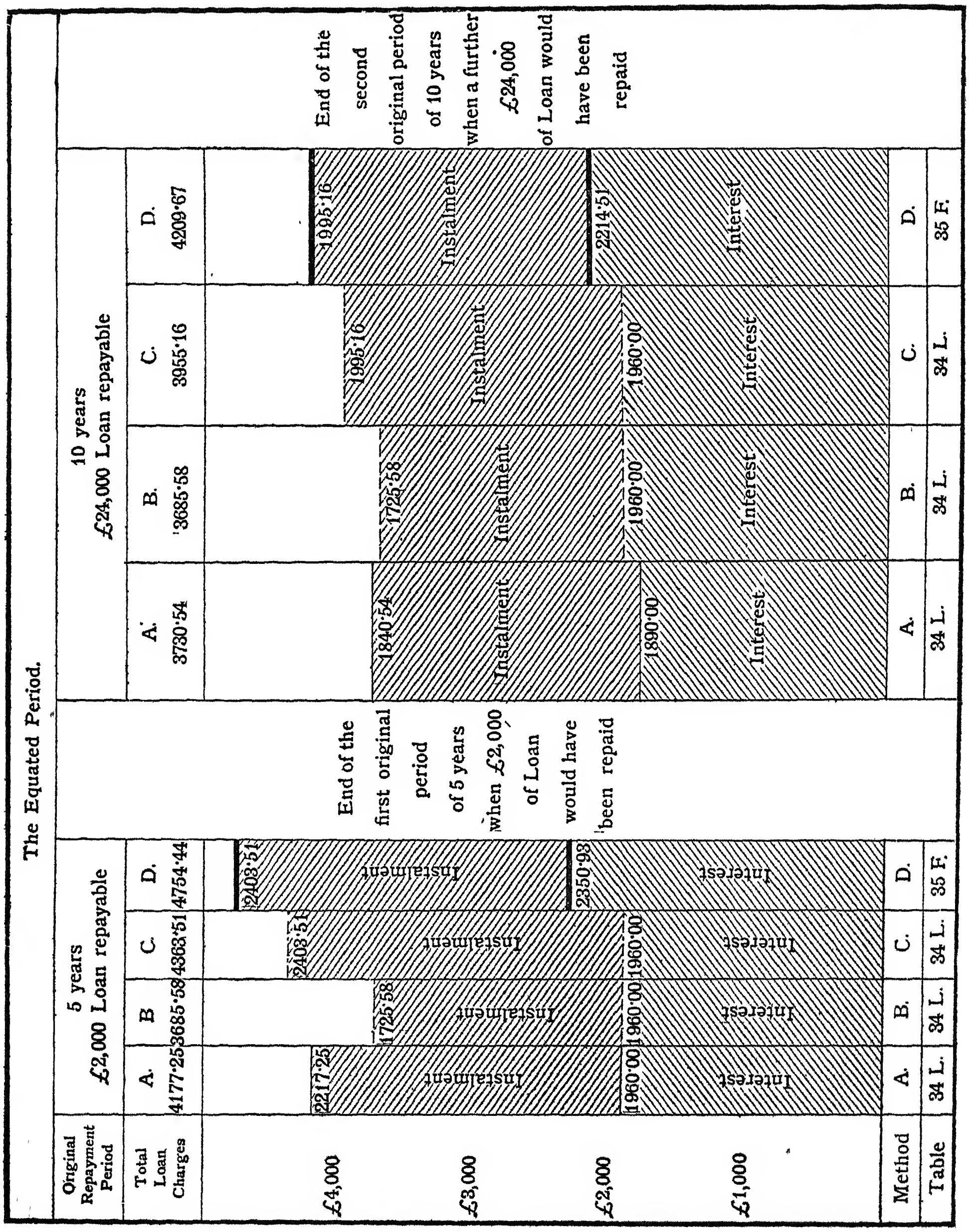
The above method of adjusting the annual incidence of the total loan burden may, and undoubtedly will, appear complicated when compared with the rough and ready method now adopted. It will certainly increase the labour involved upon the equation of the period of repayment of new loans authorised for outlays of varying natures and also upon the consolidation of existing loans, but it is sound in principle and carries out the fundamental law of local finance, that the

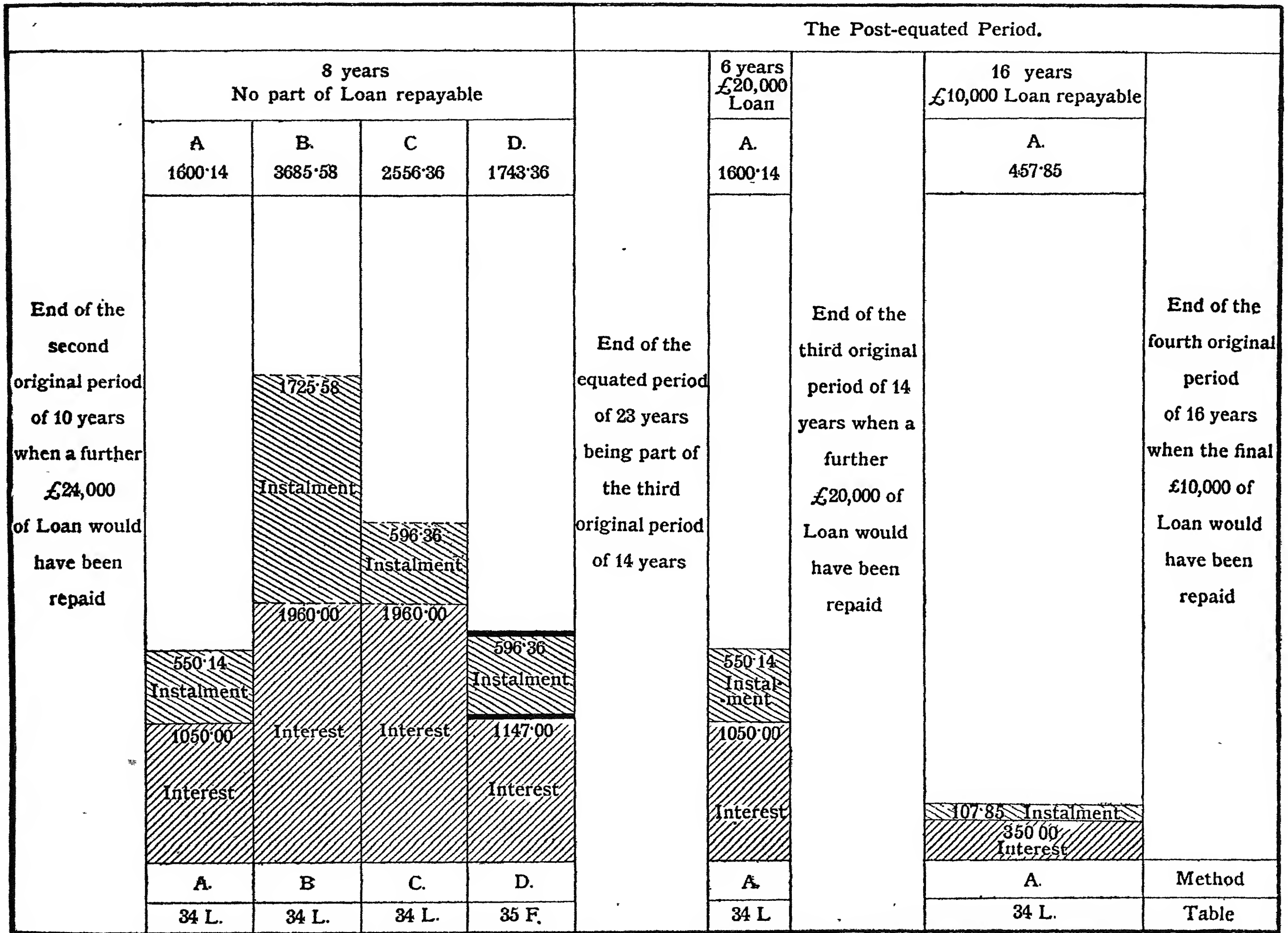
present generation shall bear at least its due burden and not transfer it to future years. It is very tempting to local leaders of finance to pose as the benefactors of the present ratepayers by adopting a method, having a high-sounding title, which has the immediate effect of reducing the present burden at the expense of the future; but it ought to be recognised, that where the only means of paying for municipal works is by annual contributions out of revenue or rate, to be spread over a prescribed period of years fixed after very careful enquiry as to the life of the asset, any reduction in the period of repayment, due solely to causes of a purely financial nature, cannot possibly equitably reduce the annual burden but must inevitably increase it. Any departure from this principle is a violation of the recognised canons of local government.

The Equation of the Incidence of Taxation.

Chart I.

A. ———————— under original conditions—before equation.
B. — — — — — — — — under equated method, as generally adopted.
C. — - — - — - — - Author's equated method (annual instalment only).
D. ━━━━━━━━ Author's equated method (annual instalment, and interest on the loan).





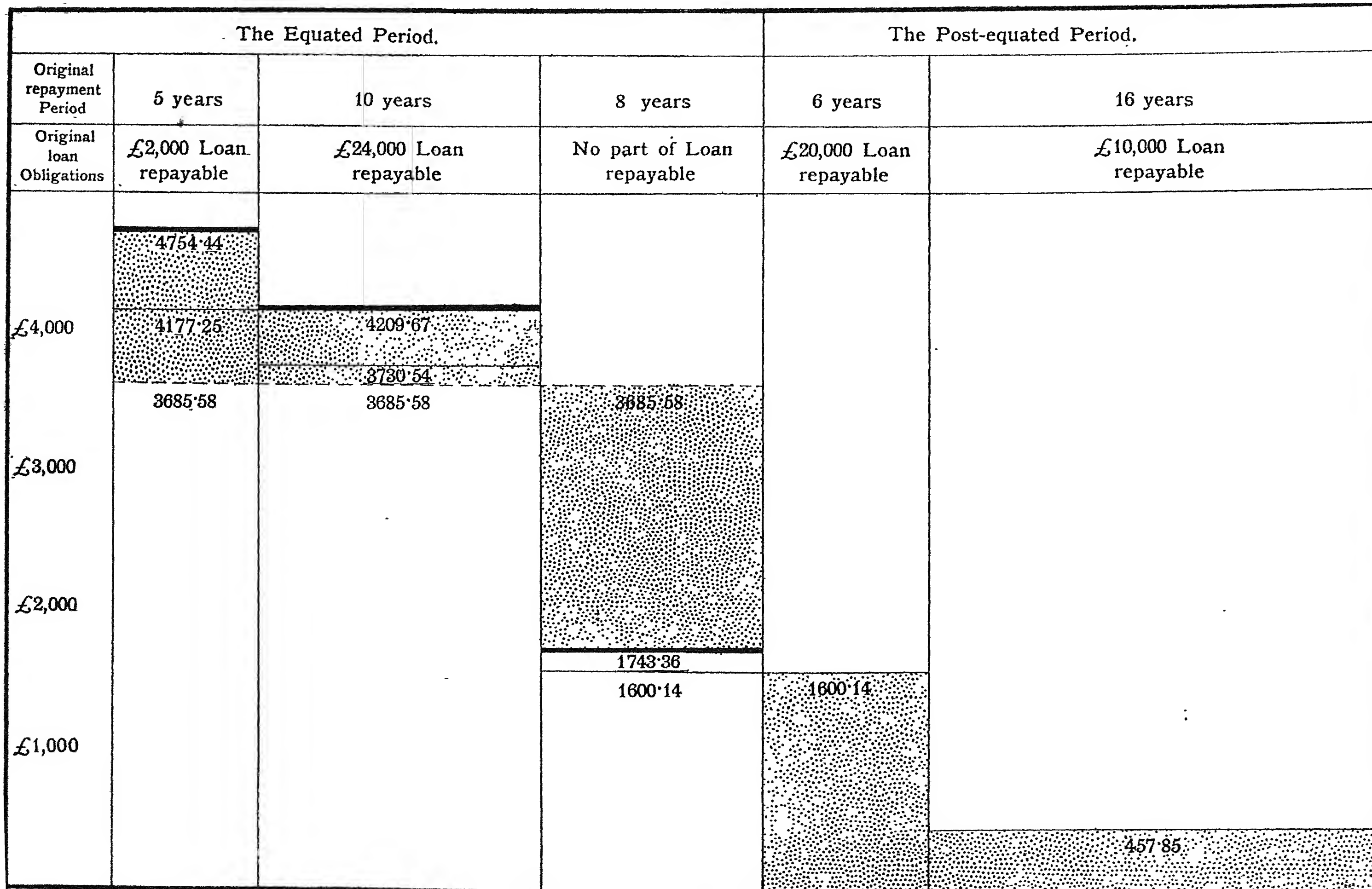
A. ———— under original conditions—before equation.

B. – – – – under equated method, as generally adopted.

C. —·—·— Author's equated method (annual instalment only).

D. ━━━━ Author's equated method (annual instalment, and interest on the loan)

Chart II.



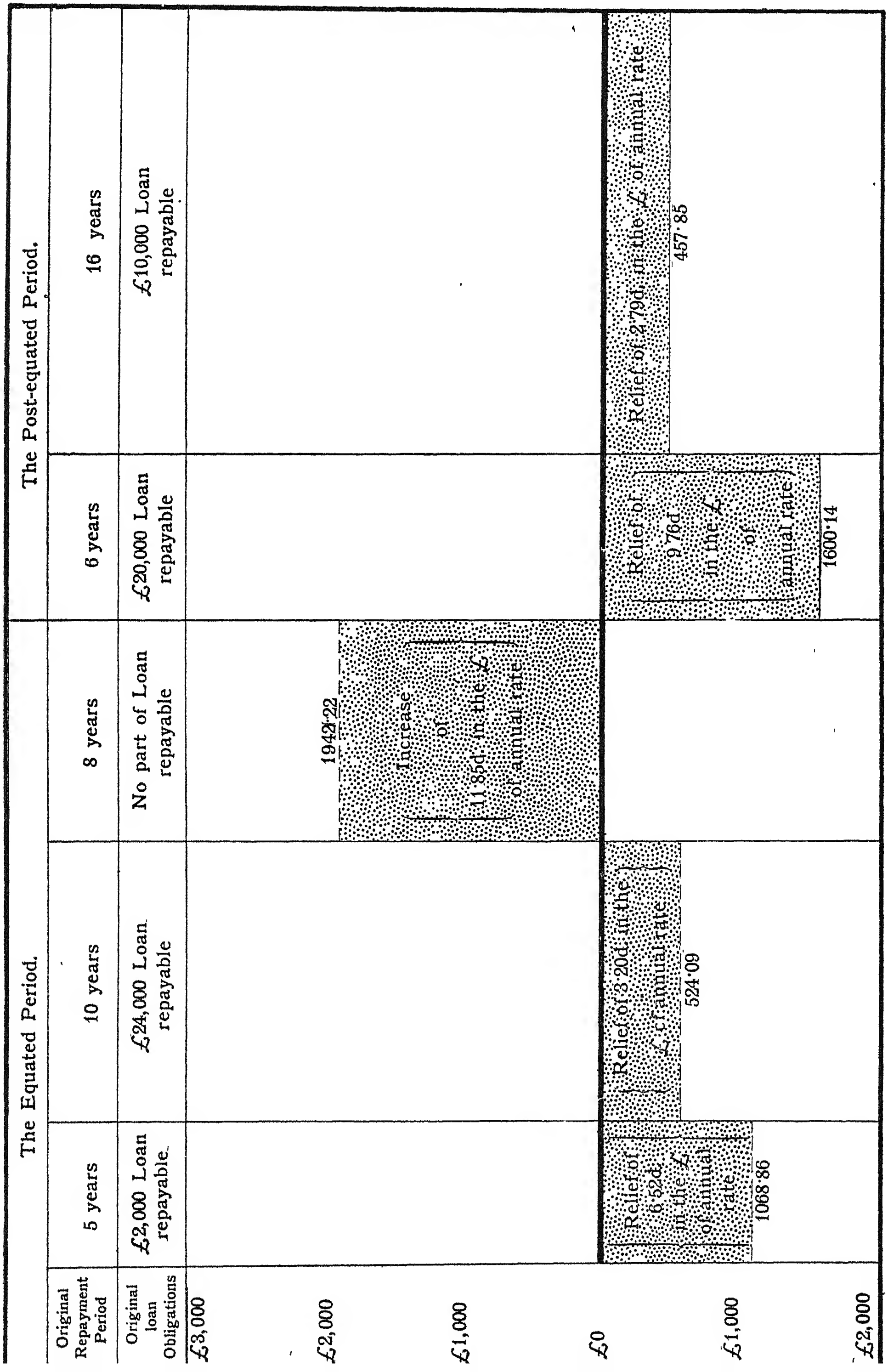
D' —— Author's equated method, (annual instalment, and interest on the loan).

B - - - - - - - - under equated method, as generally adopted.

D. ———— Author's equated method (annual instalment, and interest on the loan).

The Equation of the Incidence of Taxation.

Chart III.



APPENDIX.

Calculations Referred to in the Text

Standard Form, No. 5. **No. (XV) 3.**

To find the additional sinking fund instalment to be set aside during the unexpired portion of the repayment period, to compensate for a deficiency in the fund. Table V.

Required the annual instalment to be set aside and accumulated for 13 years at $3\frac{1}{2}$ per cent., which is equivalent to a deficiency of £469·744 in the amount now in the fund.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter VIII.

Log. Present sum	469·744	2·6718612
add Log. a^n, 13 years		8·9870474
		11·6589086
deduct 10		1·6589086
Required annual instalment		£45·5941

Standard Form, No. 1. **No. (XV) 4.**

To find the portion of original loan which will be provided by the future accumulation of the present investments representing the fund. Table I.

Required the amount of £9,463, at the end of 13 years, accumulated at $3\frac{1}{2}$ per cent.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter IV.

Log. Present sum	9463	3·9760288
add Log. R^N, 13 years...		0·1942245
		4·1702533
Required future amount		£14799·71

Standard Form, No. 3. **No. (XV) 5.**

To find the portion of original loan which will be provided by the accumulation of the original annual instalments to be set aside during the unexpired portion of the redemption period. Table III.

Required the amount of an annual instalment of £680·234, to be set aside and accumulated for 13 years at $3\frac{1}{2}$ per cent.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter VI.

Log. Annuity...	680·234	2·8326581
add Log. R^N, 13 years, +10		10·1942245
		13·0268826
deduct Log. a^n		8·9870474
		4·0398352
Required future amount		£10960·62

, No. 3x

Standard Form, No. 1. **No. (XV) 6.**

To find the portion of original loan which will be unprovided if the present deficiency in a sinking fund be allowed to remain uncorrected during the remainder of the redemption period. Table I.

Required the amount of £469·744, at the end of 13 years, accumulated at $3\frac{1}{2}$ per cent.

(B) By Table I, 13 years, $3\frac{1}{2}$ per cent. Rule 2, Chapter IV.

Log. Amount of £1	1·56395	0·1942245
add Log. present sum	469·744	2·6718612
		2·8660857
Required future amount		£734·659

Standard Form, No. 5. **No. (XVII) 1.**

To find the amount by which the annual sinking fund instalment may be reduced in consequence of a payment into the fund of proceeds of sale of part of the security for the loan.

Table V.

Required the annuity which may be purchased with £4,560 for 13 years at $3\frac{1}{2}$ per cent.

(B) By Table V, 13 years, $3\frac{1}{2}$ per cent. Rule 2, Chapter VIII.

Log. Annuity £1 will purchase ...	0·097061	$\bar{2}$·9870474
add Log. present sum	4560	3·6589648
		2·6460122
Required annuity		£442·6008

Standard Form, No. 1. **No. (XVII) 2.**

To find the portion of the original loan which will be provided by the future accumulation of the present investments representing the fund.

Table I.

Required the amount of £9932·744 at the end of [illegible] years, accumulated at $3\frac{1}{2}$ per cent.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter IV.

Log. Present sum	9932·744	3·9970693
add Log. R^N, 13 years		0·1942245
		4·1912938
Required future amount		£15534·375

Standard Form, No. 1. **No. (XVII) 3.**

To find the portion of the original loan which will be provided by the future accumulation of the proceeds of sale of assets paid into the fund. Table I.

Required the amount of £4,560 at the end of 13 years, accumulated at $3\frac{1}{2}$ per cent.

(B) By Table I, 13 years, at $3\frac{1}{2}$ per cent. Rule 2, Chapter IV.

Log. Amount of £1	1·56395	0·1942245
add Log. present sum	4560	3·6589648
		3·8531893
Required future amount		£7131·64

Standard Form, No. 3x. **No. (XVII) 4.**

To find the annual instalment which will provide the amount of loan represented by the proceeds of sale of part of the security paid into the fund. Table III.

Required the annual instalment to be set aside and accumulated at $3\frac{1}{2}$ per cent. to provide £7131·64 at the end of 13 years.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chaper XIII.

Log. Amount of loan	7131·64	3·8531893
add Log. a^n, 13 years		8·9870474
		12·8402367
deduct Log. $R^N + 10$		10·1942245
		2·6460122
Required annual instalment...		£442·6008

Standard Form, No. 3. **No. (XVII) 5.**

To find the amount of loan which will be provided by the future accumulation of the reduced annual instalment consequent upon the payment into the fund of the proceeds of sale of part of the security. Table III.

Required the amount of an annual instalment of £237·633 to be set aside and accumulated for 13 years at $3\frac{1}{2}$ per cent.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter VI.

Log. Annuity	237·633	2·3759068
add Log. R^N, 13 years, +10		10·1942245
		12·5701313
deduct Log. a^n		8·9870474
		3·5830839
Required future amount		£3823·987

Standard Form, No. 3x. **No. (XVII) 6.**

To find the future annual increment to be accumulated to provide the balance of loan not provided by the investments representing the fund. Table III.

Required the annual instalment to be set aside and accumulated at $3\frac{1}{2}$ per cent. to provide £12002·26 at the end of 13 years.

(B) By Table III, 13 years, $3\frac{1}{2}$ per cent. Rule 2, Chapter XIII.

Log. Amount of loan	12002·26	4·0792630
deduct Log. amount of £1 per ann.	16·11303	1·2071771
		2·8720859
Required annual instalment...		£744·879

Standard Form, No. 3x. **No. (XVIII) 1.**

To find the amount by which the original annual instalment may be reduced in consequence of the withdrawal of part of the loan from the operation of the fund by reason of its conversion into ordinary share capital or stock. Table III.

Required the annual instalment to be set aside and accumulated at $3\frac{1}{2}$ per cent. to provide £5,000 at the end of 13 years.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter XIII.

Log. Amount of loan	5000	3·6989700
add Log. a^n, 13 years		8·9870474
		12·6860174
deduct Log. $R^N + 10$		10·1942245
		2·4917929
Required annual instalment		£310·308

Standard Form, No. 3x. **No. (XVIII) 2.**

To find the amended annual instalment which will provide the balance of loan not provided by the future accumulation of the present investments and the original annual instalment, after withdrawal of part of the loan from the operation of the fund. Table III.

Required the annual instalment to be set aside and accumulated at $3\frac{1}{2}$ per cent. to provide £6695·30 at the end of 13 years.

(B) By Table III, 13 years $3\frac{1}{2}$ per cent. Rule 2, Chapter XIII.

Log. Amount of loan	6695·30	3·8257700
deduct Log. amount of £1 per ann.	16·11303	1·2071771
		2·6185929
Required annual instalment...		£415·520

Standard Form, No. 3x. **No. (XVIII) 3.**

To find the future annual increment to be accumulated to provide the balance of loan not provided by the present investments representing the fund. Table III.

Required the annual instalment to be set aside and accumulated at $3\frac{1}{2}$ per cent. to provide £12,032 at the end of 13 years.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter XIII.

Log. Amount of loan	12032	4·0803378
add Log. a^n, 13 years		8·9870474
		13·0673852
deduct Log. R^N+10		10·1942245
		2·8731607
Required annual instalment...		£746·725

Standard Form, No. 3. **No. (XVIII) 4.**

To find the amount which should stand to the credit of the sinking fund. Table III.

Required the amount of an annual instalment of £7,500 to be set aside and accumulated for 7 years at 3 per cent.

(B) By Table III, 7 years, 3 per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	7·6625	0·8843684
add Log. annuity	7500	3·8750613
		4·7594297
Required future amount		£57468·48

Standard Form, No. 3x. **No. (XVIII) 5.**

To find the annual instalment for an even number of years which approximates to the instalment of specified amount not found by calculation. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £150,000 at the end of 16 years.

(B) By Table III, 16 years, 3 per cent. Rule 2, Chapter XIII.

Log. Amount of loan	150,000	5·1760913
deduct Log. amount of £1 per ann.	20·1569	1·3044233
		3·8716680
Required annual instalment...		£7441·6285

Standard Form, No. 3. **No. (XVIII) 6.**

To find the amount of loan which will be provided by the instalment of stated amount at the end of the approximate period of even years. Table III.

Required the amount of an annual instalment of £7,500 to be set aside and accumulated for 16 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity	7500	3·8750613
add Log. R^N, 16 years, + 10		10·2053956
		14·0804569
deduct Log. a^n		8·9009723
		5·1794846
Required future amount		£151176·59

Standard Form, No. 3. **No. (XVIII) 7.**

To find the portion of the original loan, being the accumulation of an intentional error in the sinking fund instalment assumed in the adjustment. Table III.

Required the amount of £58·3715 per annum for 16 years at 3 per cent.

(B) By Table III, 16 years, 3 per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	20·1569	1·3044233
add Log. annuity	58·3715	1·7662008
		3·0706241
Required future amount		£1176·58

Standard Form, No. 3x. **No. (XVIII) 8.**

To find the amount by which the original annual instalment may be reduced in consequence of the withdrawal of part of the loan from the operation of the sinking fund by reason of its conversion into ordinary share capital or stock. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £45,000 at the end of 9 years.

(B) By Table III, 9 years, 3 per cent. Rule 2, Chapter XIII.

Log. Amount of loan	45000	4·6532125
deduct Log. amount of £1 per ann.	10·1591	1·0068555
		3·6463570
Required annual instalment...		£4429·523

Standard Form, No. 3. **No. (XVIII) 9.**

To find the amount which should stand to the credit of the fund. Table III.

Required the amount of an annual instalment of £7441·6285 to be set aside and accumulated for 7 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity	7441·6285	3·8716680
add Log. R^N, 7 years, +10		10·0898606
		13·9615286
deduct Log. a^n		9·2054922
		4·7560364
Required future amount		£57021·21

Standard Form, No. 5. **No. (XVIII) 10.**

To find the amount by which the annual instalment may be reduced in consequence of a surplus in the fund. Table V.

Required the annuity which may be purchased with £447·27 for 9 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VIII.

Log. Present sum	447·27	2·6505698
add. Log. a^n, 9 years		9·1086795
		11·7592493
deduct 10		1·7592493
Required annuity		£57·4446

Standard Form, No. 1. **No. (XVIII) 11.**

To find the amount of loan which will be provided by the future accumulation of the present investments of the amount in the fund. Table I.

Required the amount of £57468·48 at the end of 9 years, accumulated at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter IV.

Log. Present sum	59468·48	4·7594297
add Log. R^N, 9 years		0·1155350
		4·8749647
Required future amount		£74983·335

Standard Form, No. 3. **No. (XVIII) 12.**

To find the amount of loan which will be provided by the future accumulation of the reduced annual instalment in consequence of the withdrawal of part of the loan from the operation of the fund. Table III.

Required the amount of an annual instalment of £2954·66 to be set aside and accumulated for 9 years at 3 per cent.

(B) By Table III, 9 years, 3 per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	10·1591	1·0068555
add Log. annuity	2954·66	3·4705075
		4·4773630
Required future amount		£30016·70

Standard Form, No. 3. **No. (XVIII) 13.**

To find the amount of loan represented by the adjustment to be made in the annual instalment in consequence of an intentional error introduced for purpose of calculation. TableIII.

Required the amount of an annual instalment of £40·215 to be set aside and accumulated for 9 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity	40·215	1·6043881
add Log. R^N, 9 years, +10...		10·1155350
		11·7199231
deduct Log. a^n		9·1086795
		2·6112436
Required future amount		£408·549

Standard Form, No. 4. **No. (XVIII) 14.**

To find the present amount to be paid into the fund to compensate for the intentional error introduced for purposes of calculation. Table IV.

Required the present value of an annuity of £40·215 for 9 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VII.

Log. Annuity	40·215	1·6043881
add 10		11·6043881
deduct Log. a^n, 9 years		9·1086795
		2·4957086
Required present value		£313·118

Standard Form, No. 3x. **No. (XVIII) 15.**

To find the future annual increment to be accumulated to provide the balance of loan not provided for by the present investments representing the fund. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £47531·52 at the end of 9 years.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter XIII.

Log. Amount of loan	47531·52	4·6769818
add Log. a^n, 9 years		9·1086795
		13·7856613
deduct Log. R^N+10		10·1155350
		3·6701263
Required annual instalment		£4678·71

Standard Form, No. 3. **No. (XIX) 1.**

To find the portion of original loan which will be provided by the future accumulation of the annual income from the present investments. Table III.

Required the amount of an annual income of £347·648 to be added to the sinking fund and accumulated for 13 years at 3 per cent.

(B) By Table III, 13 years, 3 per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	15·6178	1·1936196
add Log. annuity	347·648	2·5411397
		3·7347593
Required future amount		£5429·494

Standard Form, No. 3. **No. (XIX) 2.**

To find the portion of the original loan which will be provided by the future accumulation of the original instalment. Table III.

Required the amount of an annual instalment of £680·234 to be set aside and accumulated for 13 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity	680·234	2·8326581
add Log. R^N, 13 years, +10		10·1668839
		12·9995420
deduct Log. a^n		8·9732643
		4·0262777
Required future amount		£10623·75

Standard Form, No. 3x. **No. (XIX) 3.**

To find the additional sinking fund instalment to compensate for a reduction in the rate of accumulation. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £509·02 at the end of 13 years.

(B) By Table III, 13 years, 3 per cent. Rule 2, Chapter XIII.

Log. Amount of loan	509·02	2·7067348
deduct Log. amount of £1 per ann.	15·6178	1·1936196
		1·5131152
Required annual instalment		£32·5923

Standard Form, No. 3. **No. (XIX) 4.**

To find the portion of the original loan which will be provided by the accumulation of the amended annual increment.
Table III.

Required the amount of an annual increment of £1060·474 to be added to the sinking fund and accumulated for 13 years at 3 per cent.

(B) By Table III, 13 years, 3 per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	15·61779	1·1936196
add Log. annuity	1060·474	3·0255000
		4·2191196
Required future amount		£16562·26

Standard Form, No. 3x. **No. (XIX) 5.**

To find the amended annual instalment to repay the balance of loan at the end of the period of repayment. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £16562·26 at the end of 13 years.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter XIII.

Log. Amount of loan	16562·26	4·2191196
add Log. a^n, 13 years		8·9732643
		13·1923839
deduct Log. R^N+10		10·1668839
		3·0255000
Required annual instalment...		£1060·474

Standard Form, No. 3. **No. (XX) 1.**

To find the amount of loan which will be provided by the future accumulation of the income from the present investments representing the fund. Table III.

Required the amount of an annual income of £297·984 to be added to the sinking fund and accumulated for 13 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity	297·984	2·4741929
add Log. R^N, 13 years, +10		10·1668839
		12·6410768
deduct Log. a^n		8·9732643
		3·6678125
Required future amount		£4653·85

Standard Form No. 3x. **No. (XX) 2.**

To find the additional annual instalment to provide the amount of loan unprovided for owing to a reduction in the rate of income from the present investments. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £775·64 at the end of 13 years.

(B) By Table III, 13 years, 3 per cent. Rule 2, Chapter XIII.

Log. Amount of Loan	775·64	2·8896602
deduct Log. amount of £1 per ann.	15·61779	1·1936196
		1·6960406
Required annual instalment		£49·664

Standard Form, No. 3. **No. (XX) 3.**

To find the amount of loan which will be provided by the accumulation of the future annual increment. Table III.

Required the amount of an annual increment of £1060·474 to be added to the sinking fund and accumulated for 13 years at 3 per cent.

(B) By Table III, 13 years, 3 per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	15·6178	1·1936196
add Log. annuity...	1060·474	3·0255000
		4·2191196
Required future amount		£16562·26

Standard Form, No. 3x. **No. (XX) 4.**

To find the amended annual instalment consequent upon a variation in the rate of income upon the present investments, but without any variation in the rate of accumulation. Table III.

Required the annual increment to be added to the sinking fund and accumulated at 3 per cent. to provide £16562·26 at the end of 13 years.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter XIII.

Log. Amount of loan	16562·26	4·2191196
add Log. a^n, 13 years		8·9732643
		13·1923839
deduct Log. $R^N + 10$		10·1668839
		3·0255000
Required annual instalment...		£1060·474

Standard Form, No. 3. **No. (XXI) 1.**

To find the amount of loan which will be provided by the accumulation of the annual income from the present investments under the altered conditions. Table III.

Required the amount of an annual income of £297·984 to be added to the sinking fund and accumulated for 13 years at $2\frac{1}{2}$ per cent.

(C) By Thoman's Table, $2\frac{1}{2}$ per cent. Rule 3, Chapter VI.

Log. Annuity	297·984	2·4741929
add Log. R^N, 13 years, +10		10·1394103
		12·6136032
deduct Log. a^n		8·9592717
		3·6543315
Required future amount		£4511·6094

Standard Form, No. 3. **No. (XXI) 2.**

To find the amount of loan which will be provided by the accumulation of the original annual instalment under the altered conditions. Table III.

Required the amount of an annual instalment of £680·234 to be set aside and accumulated for 13 years at $2\frac{1}{2}$ per cent.

(B) By Table III, 13 years, $2\frac{1}{2}$ per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	15·14044	1·1801386
add Log. annuity	680·234	2·8326581
		4·0127967
Required future amount		£10299·038

Standard Form, No. 3. **No. (XXI) 3.**

To find the amount of loan which will be provided by the accumulation of the additional annual instalment under the altered conditions. Table III.

Required the amount of an annual instalment of £32·592 to be set aside and accumulated for 13 years at $2\frac{1}{2}$ per cent.

(C) By Thoman's Table, $2\frac{1}{2}$ per cent. Rule 3, Chapter VI.

Log. Annuity	32·592	1·5131152
add Log. R^N, 13 years, +10		10·1394103
		11·6525255
deduct Log. a^n		8·9592717
		2·6932538
Required future amount		£493·462

Standard Form, No. 3x. **No. (XXI) 4.**

To find the annual instalment required to provide the balance of loan which will be unprovided for owing to a reduction in the rate of accumulation, etc. Table III.

Required the annual instalment to be set aside and accumulated at $2\frac{1}{2}$ per cent. to provide £1258·15 at the end of 13 years.

(B) By Table III, 13 years, $2\frac{1}{2}$ per cent. Rule 2, Chapter XIII.

Log. Amount of loan	1258·15	3·0997325
deduct Log. amount of £1 per ann.	15·14044	1·1801386
		1·9195939
Required annual instalment...		£83·0986

Standard Form, No. 3. **No. (XXI) 5.**

To prove that the amended annual increment as ascertained will complete the final repayment of the loan under the altered conditions. Table III.

Required the amount of an annual increment of £1093·909 to be added to the sinking fund and accumulated for 13 years at $2\frac{1}{2}$ per cent.

(B) By Table III, 13 years, $2\frac{1}{2}$ per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	15·14044	1·1801386
add Log. annuity	1093·909	3·0389812
		4·2191198
Required future amount		£16562·26

Standard Form, No. 1. **No. (XXIV) 1.**

To find the amount of loan which will be provided by the future accumulation of the present investments representing the fund. Table I.

Required the amount of £9932·744 at the end of 8 years, accumulated at $3\frac{1}{2}$ per cent.

(B) By Table I, 8 years, $3\frac{1}{2}$ per cent. Rule 2, Chapter IV.

Log. Amount of £1	1·31681	0·1195228
add Log. present sum	9932·744	3·9970693
		4·1165921
Required future amount		£13079·53

Standard Form, No. 3. **No. (XXIV) 2.**

To find the amount of loan which will be provided by the future accumulation of the original annual instalment. Table III.

Required the amount of an annual instalment of £680·234 to be set aside and accumulated for 8 years at $3\frac{1}{2}$ per cent.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter VI.

Log. Annuity	680·234	2·8326581
add Log. R^N, 8 years, +10		10·1195228
		12·9521809
deduct Log. a^n		9·1627932
		3·7893877
Required future amount		£6157·2614

Standard Form, No. 3x. **No. (XXIV) 3.**

To find the additional annual instalment to be set aside and added to the fund to compensate for the reduction in the redemption period. Table III.

Required the annual instalment to be set aside and accumulated at $3\frac{1}{2}$ per cent. to provide £7258·21 at the end of 8 years.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter XIII.

Log. Amount of loan	7258·21	3·8608294
add Log. a^n, 8 years		9·1627932
		13·0236226
deduct Log. R^N+10		10·1195228
		2·9040998
Required annual instalment...		£801·8624

Standard Form, No. 3. **No. (XXIV) 4.**

To find the amount of loan which will be provided by the future accumulation of the present annual income from investments for the unexpired portion of the repayment period.

Table III.

Required the amount of an annual income of £347·648 to be added to the sinking fund and accumulated for 13 years at $3\frac{1}{2}$ per cent.

(B) By Table III, 13 years, $3\frac{1}{2}$ per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	16·11303	1·2071771
add Log. annuity	347·648	2·5411397
		3·7483168
Required future amount		£5601·66

Standard Form, No. 3. **No. (XXIV) 5.**

To find the amount of loan which will be provided by the future accumulation of the present annual income from investments at the end of the substituted period of repayment.

Table III.

Required the amount of an annual income of £347·648 to be added to the sinking fund and accumulated for 8 years at $3\frac{1}{2}$ per cent.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent. Rule 3, Chapter VI.

Log. Annuity	347·648	2·5411397
add Log. R^N, 8 years, +10		10·1195228
		12·6606625
deduct Log. a^n		9·1627932
		3·4978693
Required future amount		£3146·81

Standard Form, No. 3. **No. (XXIV) 6.**

To find the amount of loan which will be provided at the end of the substituted period of repayment by the accumulation of the amended annual increment. Table III.

Required the amount of an annual increment of £1829·744 to be added to the sinking fund and accumulated for 8 years at $3\frac{1}{2}$ per cent.

(B) By Table III, 8 years, $3\frac{1}{2}$ per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	9·05169	0·9567296
add Log. annuity	1829·744	3·2623904
		4·2191200
Required future amount		£16562·26

Standard Form, No. 3. **No. (XXVI) 1.**

To find the amount of loan which will be provided by the future accumulation of the present annual instalment at the end of the substituted period of repayment. Table III.

Required the amount of an annual instalment of £712·826 to be set aside and accumulated for 8 years at $3\frac{1}{2}$ per cent.

(C) By Thoman's Table, $3\frac{1}{2}$ per cent.. Rule 3, Chapter VI.

Log. Annuity	712·826	2·8529836
add Log. R^N, 8 years, +10		10·1195228
		12·9725064
deduct Log. a^n		9·1627932
		3·8097132
Required future amount		£6452·28

Standard Form, No. 3x. **No. (XXVI) 2.**

To find the additional annual instalment to be set aside and added to the fund in consequence of a reduction in the period of repayment accompanied by an increase in the rate of accumulation. Table III.

Required the annual instalment to be set aside and accumulated at $3\frac{1}{2}$ per cent. to provide £6963·19 at the end of 8 years.

(B) By Table III, 8 years, $3\frac{1}{2}$ per cent. Rule 2, Chapter XIII.

Log. Amount of loan	6963·19	3·8428082
deduct Log. amount of £1 per ann.	9·05169	0·9567296
		2·8860786
Required annual instalment...		£769·270

Standard Form, No. 3. **No. (XXVI) 3.**

To find the amount of loan which will be provided by the accumulation of the present annual increment. Table III.

Required the amount of an annual increment of £1060·474 to be added to the sinking fund and accumulated for 8 years at 3 per cent.

(B) By Table III, 8 years, 3 per cent. Rule 2, Chapter VI

Log. Amount of £1 per annum ...	8·89234	0·9490159
add Log. annuity	1060·474	3·0255000
		3·9745159
Required future amount		£9430·09

Standard Form, No. 3. **No. (XXVI) 4.**

To find the amount of loan which will be provided by the accumulation of the amended annual instalment. Table III.

Required the amount of an annual instalment of £816·232 to be set aside and accumulated for 8 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity	816·232	2·9118125
add Log. R^N, 8 years, +10		10·1026978
		13·0145103
deduct Log. a^n		9·1536819
		3·8608284
Required future amount		£7258·19

Standard Form, No. 3x. **No. (XXVI) 5.**

To find the amount by which the annual instalment may be reduced in consequence of a surplus of loan which will be provided by an excessive annual instalment. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £126·04 at the end of 8 years.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter XIII.

Log. Amount of loan	126·04	2·1005084
add Log. a^n, 8 years		9·1536819
		11·2541903
deduct Log. R^N+10		10·1026978
		1·1514925
Required annual instalment...		£14·174

Standard Form, No. 3. **No. (XXVII) 1.**

To find the portion of original loan which will be provided by the future accumulation of the varying annual income from the present investments, being the first stage in the method by step Table III.

Required the amount of an annual income of £347·648 to be added to the sinking fund and accumulated for 8 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity...	347·648	2·5411397
add Log. R^N, 8 years, +10		10·1026978
		12·6438375
deduct Log. a^n		9·1536819
		3·4901556
Required future amount		£3091·403

Standard Form, No. 1. **No. (XXVII) 2.**

To find the accumulated amount, at the end of the unexpired period, of the amount found in the previous calculation, being the second stage in the method by step. Table I.

Required the amount of £3091·403 at the end of 5 years accumulated at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter IV.

Log. Present sum	3091·403	3·4901556
add Log. R^N, 5 years		0·0641861
		3·5543417
Required future amount		£3583·783

Standard Form, No. 3. **No. (XXVII) 3.**

To find the portion of original loan which will be provided by the future accumulation of the reduced annual income from the present investments during the second part of the unexpired repayment period. Table III.

Required the amount of an annual income of £297·984 to be added to the sinking fund and accumulated for 5 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity...	297·984	2·4741929
add Log. R^N, 5 years, +10		10·0641861
		12·5383790
deduct Log. a^n		9·3391623
		3·1992167
Required future amount		£1582·037

Standard Form, No. 3x. **No. (XXVII) 5.**

To find the additional annual instalment required in consequence of a reduction in the rate of income from investments during the later years of the unexpired period of repayment. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £263·67 at the end of 13 years.

(B) By Table III, 13 years, 3 per cent. Rule 2, Chapter XIII.

Log. Amount of loan	263·67	2·4210607
deduct Log. amount of £1 per ann.	15·6178	1·1936196
		1·2274411
Required annual instalment...		£16·8827

Standard Form, No. 3x. **No. (XXVII) 6.**

To find the equated annual income to be received over the whole of the unexpired period which is equivalent to the varying amounts of income to be received during the first and second parts of such period respectively. Table III.

Required the annual instalment to be set aside and accumulated at 3 per cent. to provide £5165·82 in 13 years.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter XIII.

Log. Amount of loan	5165·82	3·7131393
add Log. a^n, 13 years		8·9732643
		12·6864036
deduct Log. $R^N + 10$		10·1668839
		2·5195197
Required annual instalment...		£330·765

Standard Form, No. 3. **No. (XXVII) 7.**

To find the amount which will be in the fund at the end of the first part of the unexpired period of repayment, being the accumulation of the amended annual increment during that period. Table III.

Required the amount of an annual increment of £1077·357 to be added to the sinking fund and accumulated for 8 years at 3 per cent.

(B) By Table III, 8 years, 3 per cent. Rule 2, Chapter VI.

Log. Amount of £1 per annum ...	8·89234	0·9490159
add Log. annuity	1077·357	3·0323597
		3·9813756
Required future amount		£9580·22

Standard Form, No. 1. **No. (XXVII) 8.**

To find the amount of loan which will be provided at the end of the repayment period, being the accumulation during the second part of such period of the amount in the fund at the end of the first part. Table I.

Required the amount of £9580·220 at the end of 5 years accumulated at 3 per cent.

(B) By Table I, 5 years, 3 per cent. Rule 2, Chapter IV.

Log. Amount of £1	1·1593	0·0641861
add Log. present sum	9580·22	3·9813756
		4·0455617
Required future amount		£11106·10

Standard Form, No. 3. **No. (XXVII) 9.**

To find the amount of loan which will be provided by the accumulation of the amended annual increment during the second part of the unexpired repayment period. Table III.

Required the amount of an annual increment of £1027·693 to be added to the sinking fund and accumulated for 5 years at 3 per cent.

(C) By Thoman's Table, 3 per cent. Rule 3, Chapter VI.

Log. Annuity	1027·693	3·0118636
add Log. R^N, 5 years, +10		10·0641861
		13·0760497
deduct Log. a^n		9·3391623
		3·7368874
Required future amount		£5456·165

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